Weigthed estimates for the Stokes semigroup in the half-space

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Abstract

We consider the IBVP of the following Stokes problem:

$$\begin{array}{ll} u_t + \nabla\pi = \Delta u\,, & \text{in } (0,\infty)\times\mathbb{R}^n_+\,,\\ \nabla\cdot u = 0\,, & \text{in } (0,\infty)\times\mathbb{R}^n_+,\\ u\Big|_{x_n=0} = 0\,, & \\ u(0,x) = u_0(x)\,, & \end{array} \tag{1}$$

under the assumption that $u_0 \in L^p_w(\mathbb{R}^n_+)$ (with $p > n \geq 3$, $\alpha = 1 - \frac{n}{p}$), and $\nabla \cdot u_0 = 0$ in weak sense. Here, the symbol $\mathbb{R}^n_+$ denote the n-dimensional half space, i.e, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. The weighted Lebesgue space $L^p_w(\mathbb{R}^n_+)$ is a particular weighted space where $w(x)$ is a weight function defined as

$$w(x) = \prod_{j=1}^{m} |x - \bar{x}_j|^{\alpha_j}\,,$$

where $\bar{x}_j$, for $j = 1, \ldots, m$, are fixed (not necessarily distinct) points in $\mathbb{R}^n_+$ and $\displaystyle\sum_{j=1}^{m} \alpha_j = \alpha = 1 - \frac{n}{p}$.

The weighted space $L^p_w(\mathbb{R}^n_+)$ is scaling invariant. This request conditions the weight and the exponent p. We limit our consideration to these special cases in view of their applications to the Navier-Stokes equations.

In [1], we study the Navier-Stokes Cauchy problem in $n-$dimensions with initial data belonging to $L^p_\alpha(\mathbb{R}^n)$ and, relatively to the pertubations of the rest state, we generalize to the n-dimensional case the results obtained in [2]. However, in [2] is studied a result of stability of steady motions.

In particular, in [1] we analyze the Stokes problem establishing estimates for the solution in terms of the norm of the initial data in the aforementioned space. Moreover, we also derive the space-time asymptotic behavior of the solution as negative power of t and $|x|$, respectively. The order of decay (respect to both the variables) is imposed by exponet α of the weight $|x|$.
Regarding problem (1), we extend the results obtained in [1] for the linear problem to the case of the half-space, a physically more significant region. We find the results interesting. Indeed, for a more general weight finction, we show that a weak initial datum $u_0 \in L^p_\alpha(\mathbb{R}^n_+)$ can induce a regularizing effect for $u(t,x)$ in $(0,T) \times \mathbb{R}^n_+$.

Keywords: Stokes equations, weighted Lebesgue spaces.

References

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