Boundary Feedback Stabilization of Fluids in Besov Spaces of Low Regularity by Means of Finite Dimensional Controllers: $3D$ Navier-Stokes Equations and Boussinesq Systems.

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ABSTRACT

The problem of boundary-based uniform stabilization of Navier-Stokes equations was initiated in the pioneering work of A.Fursikov [F.1]-[F.3], 2001-2004. This investigation spurred intense interest in the PDE-control community and generated many works on the subject. Below, only the references to be used in the lectures can be listed.

One key problem that was left open until recently was: is it possible to feedback stabilize the $3-D$ Navier-Stokes equations with a 'minimally invasive' boundary-based, static, feedback control, which moreover is **finite dimensional?**

The first goal of the lectures will be to present the recent **affirmative** solution to this problem in [L-P-T.1].

More precisely, [L-P-T.1] shows that the $3D$-Navier-Stokes equations can be uniformly stabilized in the vicinity of an unstable equilibrium solution by means of a 'minimally' invasive, localized, boundary-based, tangential, static, feedback control strategy, which moreover is finite dimensional. The solution of the open problem on finite dimensionality in 3D required introducing an altogether new, suitable, **tight Besov space setting of low regularity** ('close' to L^3). Past literature was **entirely based on a Hilbert-space setting** and mostly employed the traditional approach based on optimal control theory with quadratic cost functional and related Riccati operator equations. However, in the $D = 3$ case, the well-posedness and stabilization analysis of the N-S non-linearity requires a Sobolev space setting just smoother than $H^{1/2}$; this then triggers compatibility conditions. References [L-T.1]-[L-T.2] provided the tangential-based uniform stabilization with **finite dimensional** feedback controllers **only for** $D = 2$. In the case D=3, the desired feedback controller was established to be finite dimensional only with Initial Conditions compactly supported. The finely tuned Besov space setting that was identified for the solution of the $3D$-open problem has to enjoy two potentially conflicting features: on the one hand, it must have a topological level sufficiently low in order not to recognize Compatibility Conditions, while at the same time the topological level must be sufficiently high to handle the $3D$- N-S non-linearity. One critical initial stumbling obstruction, following the approach of [T.1], is the issue of feedback stabilization of the **unstable finite dimensional** component via Kalman controllability algebraic rank condition. This condition is converted in establishing suitable unique continuation properties of the adjoint eigenvalue problem of the Oseen equations. This issue was solved in [L-T.1]-[L-T.2] and needed to avoid the setting of the **counter-example** in [F-L]. Another critical issue consists in establishing in the required Besov setting **maximal L^p regularity up to T=infinity** for the **boundary feedback** problem [L-P-T.3], while the intense literature on maximal regularity considers only **homogeneous boundary** problems. The new conceptual and technical difficulties of the Besov setting were first tested in [L-P-T.4] for the preliminary case of localized **interior** feedback control.

On the basis of [L-P-T.1] the next goal is to uniformly stabilize the $3D$ Boussinesq system (N-S equations coupled with a thermal equation) again with minimally invasive localized feedback control, the one for the $3D$ Navier-Stokes component being boundary-based, tangential, static, finite dimensional, as for the N-S equations alone. While we leave the solution of this problem to the next step of the investigation, in the lectures we shall present the following preliminary case solved in [L-P-T.2]. This work shows that a Boussinesq system can likewise be uniformly stabilized near an unstable equilibrium pair by a finite dimensional static, feedback control strategy in a suitable Besov setting by means of a scalar localized feedback control acting on the **boundary of the thermal component;** and a **localized interior feedback control** acting on the Navier-Stokes component, which moreover can be taken of reduced dimension (3 -1)=2 (a novelty over the Navier-Stokes equations). A preliminary contribution using localized

interior feedback controller in the required Besov setting is given in [L-P-T.5] where the N-S feedback controller may in fact be one or even two units less than the dimension of the control space, depending on geometrical conditions. The reason rest on suitable unique continuation properties of the adjoint Boussinesq problem [T-W] which again play a critical role to verify the Kalman algebraic controllability rank condition for the **finite dimensional unstable component** of the problem. For the Boussinsq problem such **adjoint problem is more amenable** than for the direct eigen-problem, unlike the case of the N-S case.

Future work intends to solve the stabilization problem for the following two problems: (1) the $3D$ Boussinesq problem with boundary based feedback control on the Navier-Stokes component and (2) the magnetohydrodynamics (MHD) system with boundary-based finite dimensional feedback controller.

Selected references to be invoked in the lectures

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