Weak solutions for the Stokes system for compressible fluids with general pressure

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Introduction

We examine the existence and uniqueness of solutions to the Stokes system for compressible fluids on the d-dimensional torus $\mathbb{T}^d$

$$\begin{cases} \varrho_t + \operatorname{div}(\varrho u) = 0, \\ -\mu\Delta u - \nabla(\lambda+\mu)\operatorname{div} u + \nabla p(\varrho) = 0, \end{cases} \tag{1}$$

where $\varrho\colon [0,T]\times\mathbb{T}^d \to \mathbb{R}$ and $u\colon [0,T]\times\mathbb{T}^d \to \mathbb{R}^d$ are the sought fluid density and velocity field. The function $p(\varrho)$ denotes the pressure term, and the parameters μ, λ represent the first and the second viscosity.

Additional assumptions and initial conditions:

- $\operatorname{rot} u = 0$, or equivalently $u = \nabla\phi$ for some $\phi\colon [0,T]\times\mathbb{T}^d \to \mathbb{R}$.
  Then $\int_{\mathbb{T}^d} u(t,x)\,\mathrm{d}x = 0$ and $-\mu\Delta u - (\lambda+\mu)\nabla(\operatorname{div} u) + \nabla p = 0$ is transformed into
  $$(\lambda+2\mu)\operatorname{div} u = p(\varrho) - \fint_{\mathbb{T}^d} p(\varrho)\mathrm{d}x$$
- initial condition: $\varrho_{|t=0} = \varrho_0 \in L^\infty(\mathbb{T}^d)$

Assumptions on the pressure: We assume that p is of class C^1 and

$$0 \le p(\varrho) \le C\varrho \int_0^\varrho \frac{p(s)}{s^2}\,\mathrm{d}s \tag{2}$$

Main result

Theorem 1. *Let $\varrho_0 \in L^\infty$ and the pressure p be of class C^1 and satisfy (2). Then there exists a unique global in time weak solution to*

$$\begin{aligned} &\varrho_t + \operatorname{div}(\varrho u) = 0, \\ &\operatorname{div} u = p(\varrho) - \fint_{\mathbb{T}^d} p(\varrho)\mathrm{d}x \end{aligned} \tag{3}$$

satisfying $\varrho_{|t=0} = \varrho_0$, $\operatorname{rot} u = 0$ and

$$\begin{aligned} &\varrho \in L^\infty([0,\infty)\times\mathbb{T}^d), \\ &u \in L^\infty([0,\infty)\times\mathbb{T}^d), \\ &\nabla u \in L^\infty([0,\infty); BMO),\ \operatorname{div} u \in L^\infty([0,\infty)\times\mathbb{T}^d). \end{aligned}$$

Outline of the proof:

- a priori estimates
- existence and uniqueness in Lagrangian coordinates
- transformation from Lagrangian to Eulerian coordinates
- uniqueness of such transformation

A priori estimates

For any $T > 0$,

$$\int_0^T \int_{\mathbb{T}^d} (\operatorname{div} u)^2 \mathrm{d}x\mathrm{d}t + \sup_{t\in[0,T]} \int_{\mathbb{T}^d} G(\varrho)\mathrm{d}x \le C \int_{\mathbb{T}^d} G(\varrho_0)\mathrm{d}x,$$

where $G(\varrho) = \varrho \int_{\bar\varrho}^{\varrho} \frac{p(s)}{s^2}\mathrm{d}s$.

As $p(\varrho) \le CG(\varrho)$, we have

$$\sup_{t\in[0,T]} \fint_{\mathbb{T}^d} p(\varrho)\mathrm{d}x \le C \sup_{t\in[0,T]} \int_{\mathbb{T}^d} G(\varrho)\mathrm{d}x \le C$$

Lagrangian coordinates

Let $x(t,y)$ be the flow of u, i.e.

$$\dot{x}(t,y) = u(t,x), \quad x(0,y) = y.$$

We put

$$\begin{aligned} \eta(t,y) &= \varrho(t, x(t,y)), \\ \sigma(t,y) &= \operatorname{div} u(t, x(t,y)). \end{aligned}$$

In the new variables the equation (3) has the form

$$\begin{aligned} &\eta_t + \eta\sigma = 0, \\ &\sigma = p(\eta) - \{p(\eta)\}_\sigma, \end{aligned} \tag{4}$$

where

$$\{p(\eta)\}_\sigma = \fint_{\mathbb{T}^d} p(\eta(t,y)) \exp\left(\int_0^t \sigma(s,y)\mathrm{d}s\right)\mathrm{d}y.$$

Global boundedness of η: assume $\eta, \sigma \in L^\infty([0,T]\times\mathbb{T}^d)$.

$$\eta_t + \eta\sigma = 0 \ \Rightarrow\ \eta(t,y) = \varrho_0(y)\exp\left(-\int_0^t \sigma(s,y)\mathrm{d}s\right)$$

$$\Rightarrow\ \eta(\cdot,y) \text{ is continuous}$$

If r is such that $p(r) > \sup_{t\in[0,T]}\{p(\eta)\}_\sigma$ and $r > \|\varrho_0\|_{L^\infty(\mathbb{T}^d)}$, then

$$\partial_t\eta_{|\eta=r} = -r(p(r) - \{p(\eta)\}_\sigma) < 0$$

$\Rightarrow \eta$ cannot exceed the value r, which does not depend on T.

The unique existence follows then from Banach fixed point theorem.

Uniqueness of solutions to (3) – preliminary observations

Goal: show that if u_1, u_2 have the regularity from Theorem 1 and satisfy

$$\operatorname{div} u_i(t, x_i(t,y)) = \sigma(t,y), \quad i = 1,2$$

where x_i is the flow of u_i, then $\|u_1 - u_2\|_{L^2} = 0$ for every t.
Having that, the uniqueness of solutions to (4) yields the uniqueness of u, moreover the classical uniqueness results for the continuity equation [4] provide the uniqueness of ϱ.

Definition of the flow x_s: for $u_1, u_2 \in L^\infty([0,T]\times\mathbb{T}^d)$ such that $\operatorname{div} u_i \in L^\infty([0,T]\times\mathbb{T}^d)$ and $\nabla u_i \in L^\infty(0,T; BMO)$ define $x_s(t,y)$ as the solution to an ODE

$$\dot{x}_s = s u_1(t, x_s) + (1-s)u_2(t,x_s), \quad x_s(0,y) = y, \quad s\in[0,1]$$

Estimates on $\frac{\mathrm{d}x_s}{\mathrm{d}s}$:

- for sufficiently small t, $\frac{\mathrm{d}x_s}{\mathrm{d}s} \in L^p$ for some $p > 4$
- differential inequality on $\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2 \le C\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2\left(1 + \left|\ln\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2\right|\right) + C\|u_1 - u_2\|_2^2 \tag{5}$$

The important tool to obtain (5) is a logarithmic inequality from [5], modified for $g \in L^q$, $q > 2$ instead of $g \in L^\infty$:

$$\begin{aligned} \left|\int_{\mathbb{T}^d} fg\,\mathrm{d}x\right| \le{}& C\|f\|_{BMO}\|g\|_{L^1} \\ &\times\left(|\ln\|g\|_{L^1}| + \ln(e + \|g\|_{L^q}) + (1 + |\ln\|g\|_{L^1}|)\|g\|_{L^q}^{\frac{q-2}{2}}\right) \end{aligned}$$

Uniqueness – final argument

By integration by parts and the change of variables, for $\xi \in L^\infty(0,T;W^{1,2})$

$$\begin{aligned} \int_{\mathbb{T}^d}(u_1-u_2)\nabla\xi\mathrm{d}x &= -\int_{\mathbb{T}^d}(\operatorname{div}u_1 - \operatorname{div}u_2)\xi\,\mathrm{d}x \\ &= -\int_{\mathbb{T}^d}\sigma(t,y)J(t,y)(\xi(t,x_1(t,y)) - \xi(t,x_2(t,y)))\mathrm{d}y, \end{aligned}$$

where $J(t,y) = e^{\int_0^t \sigma(s,y)\mathrm{d}s}$. Although $x_i(t,\cdot), i = 1,2$ is not a diffeomorphism, the last equality is justified by Lemma 3.1 from [2]. Putting $\nabla\xi = u_1 - u_2$ and using the definition of x_s, we have

$$\begin{aligned} \int_{\mathbb{T}^d}|u_1-u_2|^2\mathrm{d}x &= -\int_{\mathbb{T}^d}\sigma(t,y)J(t,y)\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\xi(t,x_s(t,y))\mathrm{d}s\mathrm{d}y \\ &= -\int_{\mathbb{T}^d}\sigma(t,y)J(t,y)\int_0^1 (u_1(t,x_s) - u_2(t,x_s))\frac{\mathrm{d}x_s}{\mathrm{d}s}\mathrm{d}s\mathrm{d}y \\ &\le \|\sigma\|_\infty\|J\|_\infty\|u_1-u_2\|_2\int_0^1\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2\mathrm{d}s. \end{aligned}$$

$$\Rightarrow \|u_1-u_2\|_2 \le C\int_0^1\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2\mathrm{d}s$$

Putting it into (5) and integrating over s, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2\mathrm{d}s \le C\int_0^1\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2\left(1 + \left|\ln\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2\right|\right)\mathrm{d}s + C\left(\int_0^1\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2\mathrm{d}s\right)^2$$

$$\Rightarrow\ \dot\alpha \le C\alpha(1 + |\ln\alpha|) \quad \text{for} \quad \alpha(t) = \int_0^1\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2\mathrm{d}s$$

As $\frac{\mathrm{d}x_s}{\mathrm{d}s}_{|t=0} = 0$, by comparison criterion $\alpha \equiv 0$ for all t.

Hence $\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2 = 0$ and therefore $\|u_1 - u_2\|_2 = 0$.

From Lagrangian to Eulerian formulation

We use the construction from [1]. First, we find the solution u_δ for mollified $\sigma_\delta = \sigma * \kappa_\delta$ with the uniform in δ estimates

$$\sup_{0\le t\le T}\|u_\delta\|_\infty + \sup_{0\le t\le T}\|u_\delta\|_{W^{1,p}} + \sup_{0\le t\le T}\|\nabla u_\delta\|_{BMO} \le C\sup_{0\le t\le T}\|\sigma\|_\infty \le C.$$

Then we pass to the limit using the Aubin-Lions lemma and the stability of the Lagangian flow from [3]:

$$\sup_{0\le t\le T}\|x(t,y) - x_\delta(t,y)\|_{L^1(\mathbb{T}^d)} \le C\left|\ln\left(\|u - u_\delta\|_{L^1([0,T]\times\mathbb{T}^d)}\right)\right|^{-1}$$

References

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