New thought on Matsumura-Nishida theory in the L_p-L_q maximal regularity framework

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Abstract

In this talk, I will talk about gloabl well-posded of Navier-Stokes equations describing the compressible, baratoropic, viscous fluid flow in a three dimensional exterior domain with non-slip boundary conditions.

Let Ω be a three dimensional exterior domain, that is the complement, Ω^c, of Ω is a bounded domain in the three dimensional Euclidean space $\mathbb{R}^3$. Let Γ be the boundary of Ω, which is a compact C^2 hypersurface. Let $\rho = \rho(x,t)$ and $\mathbf{v} = (v_1(x,t), v_2(x,t), v_3(x,t))^\top$ be respective the mass density and the velocity field, where $M^\top$ denotes the transposed M. Let $\mathfrak{p} = \mathfrak{p}(\rho)$ be the fluid pressure, which is a smooth function of $\rho > 0$ and satisfies the condition: $\mathfrak{p}'(\rho) > 0$ for $\rho > 0$. We consider the following equations given in Euler coordinates:

$$\begin{aligned} \partial_t\rho + \mathrm{div}\,(\rho\mathbf{v}) = 0 \quad & \text{in } \Omega \times (0,T), \\ \rho(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v}) + \nu\mathrm{div}\,\mathbf{v}\mathbf{I} - \mathfrak{p}(\rho)\mathbf{I}) = 0 \quad & \text{in } \Omega \times (0,T), \\ \mathbf{v}|_\Gamma = 0, \quad (\rho, \mathbf{v})|_{t=0} = (\rho_* + \theta_0, \mathbf{v}_0) \quad & \text{in } \Omega. \end{aligned} \tag{1}$$

Here, $\mathbf{D}(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^\top$ is the deformation tensor, $\mathrm{div}\,\mathbf{v} = \sum_{j=1}^3 \partial v_j/\partial x_j$, for a 3×3 matrix K with (i,j) th component K_{ij}, $\mathrm{Div}\,K = (\sum_{j=1}^3 \partial K_{1j}/\partial x_j, \sum_{j=1}^3 \partial K_{2j}/\partial x_j, \sum_{j=1}^3 \partial K_{3j}/\partial x_j)^\top$, μ and ν are two viscous constants such that $\mu > 0$ and $\mu + \nu > 0$, and ρ_* is a positive constant describing the mass density of a reference body.

Matsumura and Nishida proved that H^3_2 norm of initial data are small enough then solutions $\rho = \rho_* + \theta$ and $\mathbf{v}$ exists globally in time with

$$\begin{aligned} \theta \in C^0((0,\infty), H^3_2(\Omega)) \cap C^1((0,\infty), H^2_2(\Omega)), \quad \nabla\rho \in L_2((0,\infty), H^2_2(\Omega)), \\ \mathbf{v} \in C^0((0,\infty), H^3_2(\Omega)) \cap C^1((0,\infty), H^1_2(\Omega)), \quad \nabla\mathbf{v} \in L_2((0,\infty),, H^3_2(\Omega)). \end{aligned}$$

Here, $H^m_q = \{v \in L_q \mid \partial^\alpha_x v \in L_q(|\alpha| \leq \ell\}$ and $H^0_q = L_q$. In this talk, if $H^1_2 \cap H^1_6$ norm of initial density and $H^1_2 \cap B^1_{6,2}$ norm of inital velocity are small enough then solutions $\rho = \rho_* + \theta$ and $\mathbf{v}$ of the equations described in Lagrange coordinates exist globally in time with

$$\begin{aligned} &\theta \in H^1_2((0,\infty), H^1_6), \partial_t\theta \in L_2((0,\infty), H^1_2), \nabla\theta \in L_2((0,\infty), L_2), \ \theta \in C^0((0,\infty), L_2) \\ &\mathbf{v} \in H^1_2((0,\infty), L_6) \cap L_2((0,\infty), H^2_6), \ \partial_t\mathbf{v}, \in L_2((0,\infty), L_2), \ \nabla\mathbf{v} \in L_2((0,\infty), H^1_2), \\ &\mathbf{v} \in C^0((0,\infty), L_2). \end{aligned}$$

Main point of proof is to prove the maximal regularity with decay order of solutions to the linearized equations. To explain how to obtain this, we write equations as $\partial_t u - Au = f$ and $u|_{t=0} = u_0$ symbolically, where f is a function corresponding to nonlinear terms and A is an closed linear operator

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with domain $D(A)$. We write $u = u_1 + u_2$, where u_1 is a solution to time shifted equations: $\partial_t u_1 + \lambda_1 u_1 - Au_1 = f$ and $u_1|_{t=0} = u_0$ with some large positive number λ_1 and u_2 is a solution to compensating equations: $\partial_t u_2 - Au_2 = \lambda_1 u_1$ and $u_2|_{t=0} = 0$. Since the fundamental solutions to time shifted equations have exponential decay properties in time, u_1 has the same decay properties as these of nonlinear terms f. Moreover u_1 belongs to the domain of A for all positive time. By Duhamel principle u_2 is given by $u_2 = \lambda_1 \int_0^t T(t-s)u_1(s)\,ds$, where $\{T(t)\}_{t\geq 0}$ is a continuous analytic semigroup associated with A. By using L_p-L_q decay properties of $\{T(t)\}_{t\geq 0}$ in the interval $0 < s < t-1$ and standard estimates of C_0 analytic semigroup: $\|T(t-s)u_0\|_{D(A)} \leq C\|u_0\|_{D(A)}$ for $t-1 < s < t$, where $\|\cdot\|_{D(A)}$ denotes a domain norm, we obtain maximal L_p-L_q regularity of u_2 with decay properties. This method seems to be a new thought to prove the global wellposedness and to be applicable for many quasilinear problems of parabolic type or parabolic-hyperbolic mixture type appearing in mathematical physics.

(1) Fluid dynamics

a) Conservation equations of fluid dynamics

b) Lagrange transformation

c) Matsumura-Nishida theory

(2) Maximal L_p-L_q regularity with decay estimate for parabolic and parabolic-hyperbolic systems.

a) $\mathcal{R}$-bounded operator families and Weis operator valued Fourier multiplier theorem

b) $\mathcal{R}$ solution operators for the Stokes equations arising from the mathematical study of compressible, barotropic, viscous flows

c) L_p-L_q decay estimates for the Stokes equations.

Suggested reading:

[1] J. Prüss and G. Simonett, *Moving Interface and Quasilinear Parabolic Evolution Equations*, Birkhauser Monographs in Mathematics, 2016, ISBN: 978-3-319-27698-4, I Backgroud 1 Problems and Stragegies for (1) a)

[2] Y. Shibata, *$\mathcal{R}$ Boundedness, Maximal Regularity and Free Boundary Problems for the Navier Stokes Equations*, Mathematical Analysis of Navier-Stokes Equtions, G. P. Galdi and Y. Shibata eds. Lecture Notes in Mathematics 2254 CIME Foundation Subseries, Springer 2020, ISBN 978-3-030-36225-6, Sect. 3.3 for (1) b)

[3] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of compressible viscous and heat-conductive gases, J. Math. Kyoto Univ. **20**, 67–104 (1980) for (1) c).

[4] A. Matsumura and T. Nishida, Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, Commun. Math. Phys. **89**, 445–464 (1983). for (1) c).

[5] L. Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann. **319**, 735–758 (2001). for (2) a).

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[8] Y. Enomoto and Y. Shibata, *On the $\mathcal{R}$-Sectoriality and the Initial Boundary Value Problem for the Viscous Compressible Fluid Flow*, Funkcialaj Ekvacioj **56** (2013), 441–505 for (2) b)

[9] Y. Shibata and Y. Enomoto, *Global Existence of Classical Solutoins and Optimal Decay Rate Via the Theory of Semigroup*, Y. Giga and A. Novotny (eds), Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Springer International Publishing AG 2017, DOI 10.1007/9783-319-10151 -4_52-1 for (2) c).