An explicit threshold for the appearance of lift on the deck of a bridge

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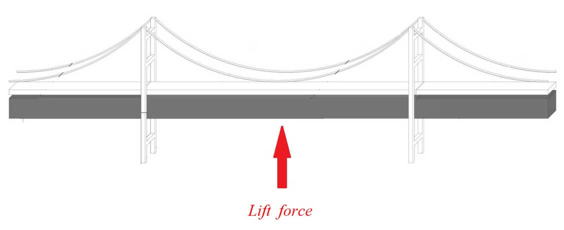
Abstract

We set up the analytical framework for studying the threshold for the appearance of a *lift force* exerted by a viscous steady fluid (the wind) on the deck of a bridge. We model this interaction as in a wind tunnel experiment, where at the inlet and outlet sections the velocity field of the fluid has a *Poiseuille flow* profile q. Since in a symmetric configuration the appearance of lift forces is a consequence of non-uniqueness of solutions, we compute an explicit threshold on the incoming flow ensuring uniqueness. This requires building an explicit solenoidal extension of the prescribed Poiseuille flow and bounding some embedding and cutoff constants.

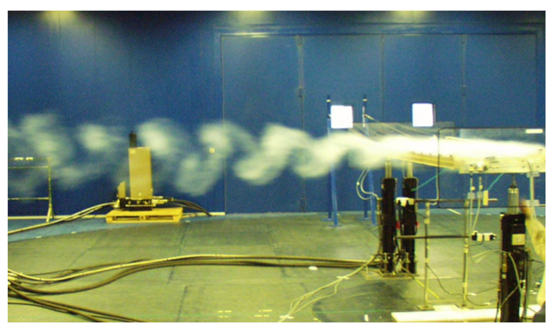
Motivation and main goal

The lift force is the component of the total force exerted by the fluid over an obstacle, *perpendicular* to the stream



Main goal: Computing an explicit threshold for the appearance of lift on the deck of a bridge.

Engineers evaluate the lift force exploiting wind tunnel tests:

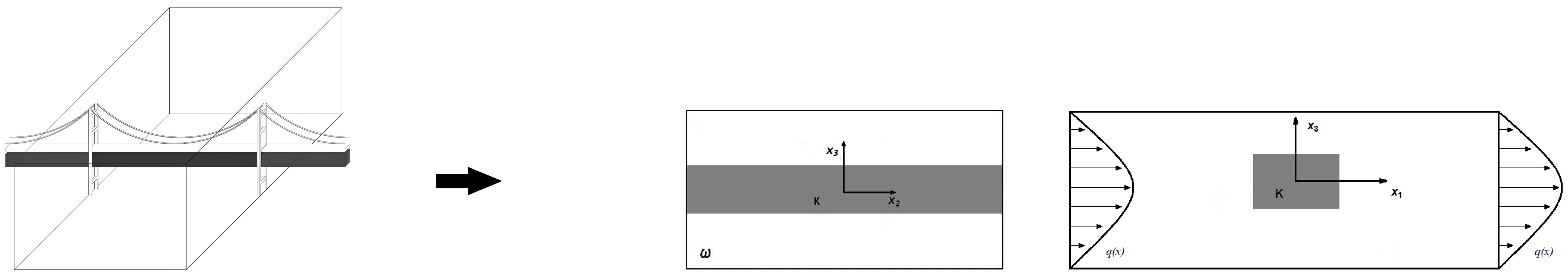


Analytical framework

$$-\mu\,\Delta u + (u\cdot\nabla)\,u + \nabla p = 0 \qquad \nabla\cdot u = 0 \qquad \text{in } \Omega, \qquad \text{(NS)}$$

$$u = q \quad \text{on } \partial T \qquad u = 0 \quad \text{on } \partial K, \qquad \text{(BC)}$$

$$T = (-L,L)\times\omega, \quad \omega = (-d,d)\times(-1,1), \quad K = (-l,l)\times(-1,1)\times(-h,h), \qquad \Omega = T\setminus\bar K \quad \text{(D)}$$



Main purpose: Analyzing the well-posedness of (NS)-(BC) $\Rightarrow$ obtaining *explicit bounds* for the **uniqueness** of its solutions. Indeed, in a *symmetrical framework*:

$$\text{uniqueness} \implies \text{no } \textit{lift} \text{ exerted over } K.$$

BC and weak formulation

At the inlet and outlet sections of the cylinder the flow is of *Poiseuille-type*:

$$q(x) = \{v_1(x_2,x_3),0,0\},\ v_1 \text{ s.t. } \|\nabla q\|_{L^2(\omega)} = k_p\,.$$

Functional space

$$V(\Omega) = \{\phi \in H^1_0(\Omega) \,|\, \nabla\cdot\phi = 0 \text{ in } \Omega\}. \qquad (1)$$

Standard trilinear form:

$$\psi\,(u,v,w) = \int_\Omega (u\cdot\nabla)v\cdot w. \qquad (2)$$

Definition: A vector field $u:\Omega\to R^3$ is called a *weak solution* to (NS)-(BC) if $u\in H^1(\Omega), u=0$ on $\partial K, \nabla\cdot u = 0$, u satisfies (BC) in the trace sense and

$$\mu(\nabla u,\nabla\phi)_{L^2(\Omega)} + \psi(u,u,\phi) = 0 \qquad \forall\,\phi\in V(\Omega).$$

Statement of the main result and steps of the proof

Main theorem: For any $k_p > 0$, there exists a weak solution u of (NS)-(BC). Moreover, there exists $\bar k_p = \bar k_p(\mu,L,d,l,h)$ such that, if

$$0 < k_p < \bar k_p(\mu,L,d,l,h)$$

then the weak solution is unique. Hence, in order to observe a lift force over the obstacle, it must be $k_p > \bar k_p$.

Steps of the proof

- Explicit expressions and bounds
  1. Determination of the **solenoidal extension**: bounds
  2. Bounds for the **Sobolev embedding constants**
- Actual proof
  1. **Existence** and **uniqueness**
  2. **Explicit threshold** for the appearance of the lift

Explicit bounds

1. **Solenoidal extension** Classical procedure by O.A. Ladyzhenskaya:

$$a(x) = \nabla\times\big(b(x)\,\theta(x)\big)\,, \qquad (3)$$

where

- $\nabla\times(b(x)) = q(x)$
- $\theta(x)$ is a C^1 cut-off function equal to 1 at all points of Ω far away from ∂K and to 0 near ∂K

Then

$$\|a\|_{L^4(\Omega)} \le \Lambda_1 k_p, \qquad \|\nabla a\|_{L^2(\Omega)} \le \Lambda_2 k_p,$$

where Λ_1 and Λ_2 are two explicit constants (see [1] for their definition).

2. **Bounds for the Sobolev embedding constants** For any $v\in H^1_0(\Omega)$, one has

$$\mathcal{S}^*\,\|v\|^2_{L^4(\Omega)} \le \|\nabla v\|^2_{L^2(\Omega)}$$

with

$$\mathcal{S}^* = \max\left\{\pi^3\left(1+\frac{1}{L^2}+\frac{1}{d^2}\right)^{1/2},\ \sqrt[3]{\frac{4\,\pi^{10}}{3\,(Ld-lh)}}\right\}^{1/2}$$

Tools to obtain $\mathcal{S}^*$: Poincaré constant in T, Faber-Krahn inequality, optimal Gagliardo-Niremberg inequality (del Pino, Dolbeault (2002)).

Actual proof

We define

$$u = \hat u + a,$$

where $\hat u$ obeys to (NS-hom):

$$-\mu\,\Delta\hat u + (\hat u\cdot\nabla)\,\hat u + (\hat u\cdot\nabla)\,a + (a\cdot\nabla)\,\hat u + \nabla p = f := \mu\,\Delta a - (a\cdot\nabla)\,a \qquad \nabla\cdot\hat u = 0 \quad \text{in } \Omega,$$

$$\hat u = 0 \quad \text{on } \partial\Omega$$

- **Uniqueness**: it relies on some *a priori* bound on $\|\nabla\hat u\|_{L^2(\Omega)}$ depending only on the data

From [2], we obtain an explicit bound for the Sobolev embedding constant in the 2D rectangular cross section ω

$$\sigma_*\,\|u\|^2_{L^4(\omega)} \le \|\nabla u\|^2_{L^2(\omega)} \qquad \text{with} \quad \sigma_* = \sqrt{3}\left(\frac{\pi}{2}\right)^{3/2}\frac{\sqrt{1+d^2}}{d} \qquad \forall\,u\in H^1_0(\omega).$$

Lemma (a priori bound): Let $\hat u$ be a weak solution of (NS-hom). If $k_p < \frac{\mu\,\sigma_*}{\mathcal{S}_*+\Lambda_2\sigma_*}$, then the following a priori estimate holds:

$$\|\nabla\hat u\|_{L^2(\Omega)} \le \frac{\mu\,\Lambda_2 k_p + \Lambda_1\frac{\Lambda_2}{\sqrt{\mathcal{S}_*}}k_p^2}{\mu - \frac{k_p}{\sigma_*} - \frac{k_p\Lambda_2}{\mathcal{S}_*}}.$$

Proposition: There exists at least one weak solution u to problem (NS)-(BC), with corresponding $p\in L^2(\Omega)$. Moreover, if

$$k_p < \bar k_p := \mu\sigma_*\frac{2\mathcal{S}_* + \sqrt{\mathcal{S}_*}\Lambda_1\sigma_* + 2\Lambda_2\sigma_* - \sigma_*\sqrt{(\sqrt{\mathcal{S}_*}\Lambda_1 + 2\Lambda_2)^2 + \frac{4\mathcal{S}_*\Lambda_2}{\sigma_*}}}{2\mathcal{S}_* + 2\sqrt{\mathcal{S}_*}\Lambda_1\sigma_* + 2\Lambda_2\sigma_*}$$

then the weak solution is unique and the fluid exerts no lift on the obstacle K.

Remark: The expression for $\bar k_p$ is fully explicit.

References

[1] F. Gazzola, C. Patriarca. An explicit threshold for the appearance of lift on the deck of bridge, preprint. 2020.

[2] F.Gazzola, G. Sperone. Steady Navier-Stokes equations in planar domains with obstacle and explicit bounds for its unique solvability, Arch. Ration. Mech. Anal. 2020.