Justification of a 2D Mathematical Model of a Flow of a Viscous Incompressible Fluid through a Turbine

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Abstract

The 3D problem of a flow of a viscous incompressible fluid through a rotating turbine wheel is reduced to an appropriate 2D problem. We describe the reduction and provide arguments for its justification.

Introduction

A rotating blade machine, as a part of a turbine or a compressor, is one of important tools of industrial mechanical engineering. Mathematical studies of flows through the blade machine are therefore of high interest, and this concerns both purely theoretical investigation and numerical simulations. As the complete mathematical modeling of such flows is mostly difficult due to the 3D geometry and usually a complicated shape of the blade machine, mathematical models are often reduced to two spatial dimensions. It should be noted that the 2D model is often mostly used and studied without a deeper justification of reduction of the originally 3D mathematical model to the 2D case. The goal of this paper is to fill in this gap and to provide arguments, justifying the aforementioned approach.

The turbine wheel and its expansion into a 2D profile cascade

We consider a turbine wheel that consists of a family of N profiles, repeating periodically in the angular direction about the axis of the wheel with the angular period $2\pi/N$. Assume that r, φ, w is a cylindrical coordinate system, attached to the wheel, such that the w–axis coincides with the axis of the wheel. As the turbine generally rotates about its axis, we call r, φ, w the *rotating frame*. The profiles are supposed to have such shapes that their intersection with the cylindrical surface $r = const.$

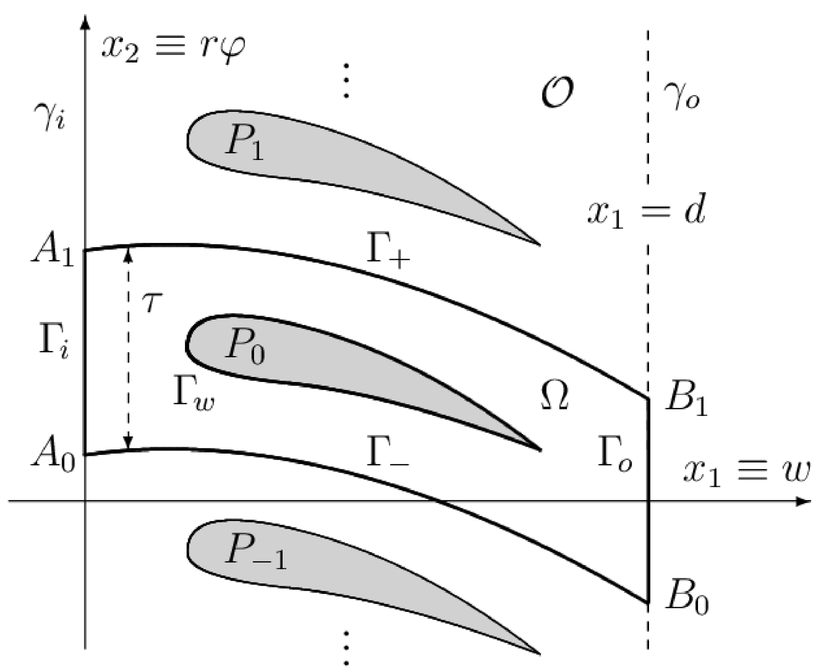

Fig. 1: The profile cascade with marked one spatial period

(for fixed r in the range $r_1 < r < r_2$) represents a periodic profile cascade that, expanded to the plane, has the form sketched on Fig. 1. Here, we use the planar Cartesian coordinate system x_1, x_2 such that $x_1 = w$ and $x_2 = r\varphi$ (for fixed $r > 0$). Since the profiles repeat periodically with the spatial period $\tau\mathbf{e}_2$, where $\tau = 2\pi r/N$ and $\mathbf{e}_2 = (0,1)$, we may assume that they create an infinite family $\{P_k\}_{k\in\mathbb{Z}}$ such that P_k are closed bounded sets in the stripe $\mathfrak{S} := \{\mathbf{x} \equiv (x_1,x_2) \in \mathbb{R}^2;\ 0 < x_1 < d\}$, with Lipschitzian boundaries, such that $P_k = P_0 + k\tau\mathbf{e}_2$ and $P_k \cap P_{k+1} = \emptyset$ for $k \in \mathbb{Z}$. Put $\mathcal{O} := \mathfrak{S} \smallsetminus \mathfrak{P}$, where $\mathfrak{P} := \cup_{k\in\mathbb{Z}} P_k$. The boundary of $\mathcal{O}$ consists of the line $x_1 = 0$ (denoted by γ_i), the line $x_1 = d$ (denoted by γ_o) and the boundary of $\mathfrak{P}$.

Equations of motion

Let us denote by r', φ', w' a cylindrical coordinate system in $\mathbb{R}^3$, attached to the fixed external observer's frame, such that the w'–axis coincides with the w–axis (i.e. with the axis of the turbine). Denote by $v_{r'}$, $v_{\varphi'}$, $v_{w'}$ the cylindrical components of the velocity of the flow of a viscous incompressible fluid through the turbine and by p' the associated pressure in the coordinate system r', φ', w'. The velocity and the pressure satisfy the Navier–Stokes equations (1)–(3) (expressing the conservation of momentum) and the equation of continuity (4) (expressing the conservation of mass):

$$\partial_t v_{r'} + v_{r'}\,\partial_{r'} v_{r'} + \frac{v_{\varphi'}}{r'}\,\partial_{\varphi'} v_{r'} + v_{w'}\,\partial_{w'} v_{r'} - \frac{v_{\varphi'}^2}{r'} + \partial_{r'} p' = f_{r'} + \nu\Big(\partial_{r'}^2 v_{r'} + \frac{1}{r'^2}\,\partial_{\varphi'}^2 v_{r'} + \partial_{w'}^2 v_{r'} + \frac{1}{r'}\,\partial_{r'} v_{r'} - \frac{2}{r'^2}\,\partial_{\varphi'} v_{\varphi'} - \frac{v_{r'}}{r'^2}\Big), \tag{1}$$

$$\partial_t v_{\varphi'} + v_{r'}\,\partial_{r'} v_{\varphi'} + \frac{v_{\varphi'}}{r'}\,\partial_{\varphi'} v_{\varphi'} + v_{w'}\,\partial_{w'} v_{\varphi'} + \frac{v_{r'}\,v_{\varphi'}}{r'} + \frac{1}{r'}\,\partial_{\varphi'} p' = f_{\varphi'} + \nu\,\Big(\partial_{r'}^2 v_{\varphi'} + \frac{1}{r'^2}\,\partial_{\varphi'}^2 v_{\varphi'} + \partial_{w'}^2 v_{\varphi'} + \frac{1}{r'}\,\partial_{r'} v_{\varphi'} + \frac{2}{r'^2}\,\partial_{\varphi'} v_{r'} - \frac{v_{\varphi'}}{r'^2}\Big), \tag{2}$$

$$\partial_t v_{w'} + v_{r'}\,\partial_{r'} v_{w'} + \frac{v_{\varphi'}}{r'}\,\partial_{\varphi'} v_{w'} + v_{w'}\,\partial_{w'} v_{w'} + \partial_{w'} p' = f_{w'} + \nu\,\Big(\partial_{r'}^2 v_{w'} + \frac{1}{r'^2}\,\partial_{\varphi'}^2 v_{w'} + \partial_{w'}^2 v_{w'} + \frac{1}{r'}\,\partial_{r'} v_{w'}\Big), \tag{3}$$

$$\frac{1}{r'}\,\partial_{r'}(r' v_{r'}) + \frac{1}{r'}\,\partial_{\varphi'} v_{\varphi'} + \partial_{w'} v_{w'} = 0, \tag{4}$$

see e.g. [4, p. 793]. As the turbine rotates, the domain between the profiles, filled by the moving fluid, changes its form in dependence on time. We denote by ω the angular speed of rotation and we assume that $\omega = const.$ In order to obtain a problem in a time-independent domain, we use the transformation $r' = r$, $\varphi' = \varphi + \omega t$, $w' = w$ and

$$\begin{aligned} v_{r'}(r',\varphi',w',t) &= v_r(r,\varphi,w,t) &&= v_r(r',\varphi'-\omega t,w',t),\\ v_{\varphi'}(r',\varphi',w',t) &= v_\varphi(r,\varphi,w,t) + r\omega &&= v_\varphi(r',\varphi'-\omega t,w',t) + r\omega,\\ v_{w'}(r',\varphi',w',t) &= v_w(r,\varphi,w,t) &&= v_w(r',\varphi'-\omega t,w',t),\\ p'(r',\varphi',w',t) &= p(r,\varphi,w,t) &&= p(r',\varphi'-\omega t,w',t). \end{aligned}$$

Note that the cylindrical coordinate system r, φ, w is attached to the rotating turbine, which means that the turbine has a fixed time-independent position in this system and v_r, v_φ, v_w are cylindrical components of the velocity of motion of the fluid, relative to the rotating frame r, φ, w. Transforming the equations (3)–(6) to the frame r, φ, w, we get

$$\partial_t v_r - \omega\,\partial_\varphi v_r + v_r\,\partial_r v_r + \frac{v_\varphi + r\omega}{r}\,\partial_\varphi v_r + v_w\,\partial_w v_r - \frac{(v_\varphi + r\omega)^2}{r} + \partial_r p = f_r + \nu\Big(\partial_r^2 v_r + \frac{1}{r^2}\,\partial_\varphi^2 v_r + \partial_w^2 v_r + \frac{1}{r}\,\partial_r v_r - \frac{2}{r^2}\,\partial_\varphi v_\varphi - \frac{v_r}{r^2}\Big), \tag{5}$$

$$\partial_t v_\varphi - \omega\,\partial_\varphi v_\varphi + v_r\,\partial_r v_\varphi + \frac{v_\varphi + r\omega}{r}\,\partial_\varphi v_\varphi + v_w\,\partial_w v_\varphi + \frac{v_r\,v_\varphi}{r} + \omega v_r + \frac{1}{r}\,\partial_\varphi p = f_\varphi + \nu\,\Big(\partial_r^2 v_\varphi + \frac{1}{r^2}\,\partial_\varphi^2 v_\varphi + \partial_w^2 v_\varphi + \frac{1}{r}\,\partial_r(v_\varphi + r\omega) + \frac{2}{r^2}\,\partial_\varphi v_r - \frac{v_\varphi + r\omega}{r^2}\Big), \tag{6}$$

$$\partial_t v_w - \omega\,\partial_\varphi v_w + v_r\,\partial_r v_w + \frac{v_\varphi + r\omega}{r}\,\partial_\varphi v_w + v_w\,\partial_w v_w + \partial_w p = f_w + \nu\,\Big(\partial_r^2 v_w + \frac{1}{r^2}\,\partial_\varphi^2 v_w + \partial_w^2 v_w + \frac{1}{r}\,\partial_r v_w\Big), \tag{7}$$

$$\frac{1}{r}\,\partial_r(r\,v_r) + \frac{1}{r}\,\partial_\varphi v_\varphi + \partial_w v_w = 0, \tag{8}$$

where $f_{r'}(r', \varphi', w', t) = f_r(r, \varphi, w, t)$, $f_{\varphi'}(r', \varphi', w', t) = f_\varphi(r, \varphi, w, t)$ and $f_{w'}(r', \varphi', w', t) = f_w(r, \varphi, w, t)$. The new terms $-2\omega v_\varphi$ and ωv_r in equations (5) and (6), respectively, are components of the Coriolis force and the term $-\omega^2 r$ in equation (5) comes from the centrifugal force. Assume that

$$v_r = 0 \tag{9}$$

and v_φ and v_w do not explicitly depend on time (i.e. the flow is steady in the rotating frame). Due to this condition, the fluid flows separately on cylindrical surfaces $r = const.$ with no exchange of mass between different surfaces. (Each streamline is a subset of a cylinder $r = c$ for some $c > 0$.) Put

$$x_1 := w, \qquad x_2 := r\varphi,$$
$$\widetilde{v}_1(r, x_1, x_2) := v_w(r, \varphi, w), \qquad \widetilde{v}_2(r, x_1, x_2) := v_\varphi(r, \varphi, w)$$

and $\widetilde{p}(r, x_1, x_2) := p(r, \varphi, w)$. The system (5)–(8) now transforms to

$$-\frac{(\widetilde{v}_2 + r\omega)^2}{r} + \partial_r \widetilde{p} = \widetilde{f}_r - \frac{2\nu}{r}\, \partial_2 \widetilde{v}_2, \tag{10}$$

$$\widetilde{v}_2\, \partial_2 \widetilde{v}_2 + \widetilde{v}_1\, \partial_1 \widetilde{v}_2 + \partial_2 \widetilde{p} = \widetilde{f}_2 + \nu \left(\partial_r^2 \widetilde{v}_2 + \partial_2^2 \widetilde{v}_2 + \partial_1^2 \widetilde{v}_2 + \frac{1}{r}\, \partial_r \widetilde{v}_2 - \frac{\widetilde{v}_2}{r^2} \right), \tag{11}$$

$$\widetilde{v}_2\, \partial_2 \widetilde{v}_1 + \widetilde{v}_1\, \partial_1 \widetilde{v}_1 + \partial_1 \widetilde{p} = \widetilde{f}_1 + \nu \left(\partial_r^2 \widetilde{v}_1 + \partial_2^2 \widetilde{v}_1 + \partial_1^2 \widetilde{v}_1 + \frac{1}{r}\, \partial_r \widetilde{v}_1 \right), \tag{12}$$

$$\partial_2 \widetilde{v}_2 + \partial_1 \widetilde{v}_1 = 0. \tag{13}$$

Applying the arguments, based on the dimensional analysis, one can neglect some terms on the right hand sides of (11) and (12). Concretely, since the changes of $\widetilde{v}_2$ and $\widetilde{v}_1$ in the r–direction (i.e. along the turbine blade) are negligible in comparison with the changes in the x_2 and x_1 directions, we neglect the terms $\partial_r^2 \widetilde{v}_2$, $r^{-1}\, \partial_r \widetilde{v}_2$, $\partial_r^2 \widetilde{v}_1$ and $r^{-1}\, \partial_r \widetilde{v}_1$. Furthermore, if we denote by V the characteristic speed, by δ the characteristic length in the x_1– or x_2–directions and by L the characteristic length in the r–direction then we deduce that the sum $\partial_2^2 \widetilde{v}_2 + \partial_1^2 \widetilde{v}_2$ on the right hand side of (12) is of the order V/δ^2, while the term $\widetilde{v}_2/r^2$ is of the order V/L^2. One can assume that $L >> \delta$, because the flow changes much more rapidly in the x_1– and x_2–directions than in the r–direction. Then the term $\widetilde{v}_2/r^2$ is negligible in comparison with $\partial_2^2 \widetilde{v}_2 + \partial_1^2 \widetilde{v}_2$, too. (These arguments are analogous to the arguments, used e.g. in the derivation of the boundary layer equations.) Thus, the equations (11), (12) take the form

$$\widetilde{v}_2\, \partial_2 \widetilde{v}_2 + \widetilde{v}_1\, \partial_1 \widetilde{v}_2 + \partial_2 \widetilde{p} = \widetilde{f}_2 + \nu \left(\partial_2^2 \widetilde{v}_2 + \partial_1^2 \widetilde{v}_2 \right), \tag{14}$$

$$\widetilde{v}_2\, \partial_2 \widetilde{v}_1 + \widetilde{v}_1\, \partial_1 \widetilde{v}_1 + \partial_1 \widetilde{p} = \widetilde{f}_1 + \nu \left(\partial_2^2 \widetilde{v}_1 + \partial_1^2 \widetilde{v}_1 \right). \tag{15}$$

Denoting $\mathbf{v} := (\widetilde{v}_1, \widetilde{v}_2)$, $\mathbf{f} = (\widetilde{f}_1, \widetilde{f}_2)$, $\nabla := (\partial_1, \partial_2)$, $\Delta := \partial_1^2 + \partial_2^2$ and $\operatorname{div} \mathbf{v} := \nabla \cdot \mathbf{v}$, we can write the equations (14), (15), (13) in the form

$$\mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mathbf{f} + \nu \Delta \mathbf{v}, \tag{16}$$

$$\operatorname{div} \mathbf{v} = 0. \tag{17}$$

This is a 2D Navier–Stokes system for fixed $r > 0$. Although $\mathbf{v}$ and p may also depend on r, it is logical to neglect this dependence and to treat $\mathbf{v}$ and p as functions of the only variables x_1, x_2.

Equation (10) reduces to the form

$$\partial_r \widetilde{p} = \widetilde{f}_r + \frac{(\widetilde{v}_2 + r\omega)^2}{r} - \frac{2\nu}{r}\, \partial_2 \widetilde{v}_2. \tag{18}$$

One cannot generally expect this equation to be satisfied in the whole 3D flow field, which is mainly caused by the assumption (10), which remarkably simplifies the system (5)–(8). Nevertheless, independently of (16) and (17), equation (18) can be considered to be a formula, providing an additional information on the rate of change of $\widetilde{p}$ in the direction parallel to the turbine blade, i.e. in the direction of the local coordinate axis r.

Conclusions

Rotating blade machines of various shapes play an important role in engineering practice. Mathematical modeling of flows through such machines is very complicated, especially due to the complicated geometry of the whole machine and the rotation. This is the reason why the mathematical model, describing the complete 3D situation, is often being reduced to an appropriate 2D mathematical model, which is much simpler. In this paper, we provide the description and justification of such a reduction.

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