ANALYSIS OF A 3D NONLINEAR, MOVING BOUNDARY PROBLEM DESCRIBING FLUID-MESH-SHELL INTERACTION

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Introduction

We consider a *nonlinear, moving boundary, fluid-structure interaction problem* between a time-dependent incompressible, viscous fluid flow, and an elastic structure composed of a cylindrical shell supported by a mesh of elastic rods. The fluid flow is modeled by the *time-dependent Navier-Stokes equations* in a three-dimensional cylindrical domain, while the lateral wall of the cylinder is modeled by the two-dimensional *linearly elastic Koiter shell equations* coupled to a one-dimensional *hyperbolic balance laws* defined on a graph domain, modeling a mesh of elastic curved rods. Two-way coupling based on kinematic and dynamic coupling conditions is assumed between the fluid and composite structure, and between the mesh of curved rods and Koiter shell. We prove the *existence of a weak solution* to this nonlinear, moving boundary FSI problem.

Motivation

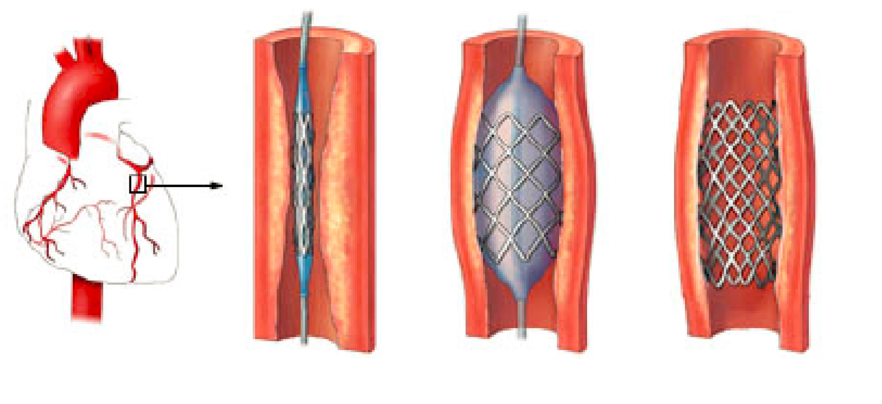

Figure 1. Coronary artery treated with a vascular stent

The fluid

Let $\Omega = \{(z,x,y) \in \mathbb{R}^3 : z \in (0,L), \sqrt{x^2+y^2} \le R\}$ and let us denote by

$$\Omega^{\boldsymbol{\eta}}(t) = \boldsymbol{\phi}^{\boldsymbol{\eta}}(t,\Omega) \text{ and } \Gamma^{\boldsymbol{\eta}}(t) = \boldsymbol{\phi}^{\boldsymbol{\eta}}(t,\Gamma)$$

the deformed fluid domain at time t, and the corresponding deformed lateral boundary, respectively, where $\boldsymbol{\eta}$ denotes the lateral boundary displacement, and $\boldsymbol{\phi}^{\boldsymbol{\eta}}$ is an arbitrary, injective and orientation preserving mapping that describes the fluid domain deformation. The time-dependent Navier-Stokes equations are used to model the flow in $\Omega^{\boldsymbol{\eta}}(t)$:

$$\left.\begin{aligned} \rho_F\left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= \nabla\cdot\boldsymbol{\sigma}, \\ \nabla\cdot\mathbf{u} &= 0, \end{aligned}\right\} \text{ in } \Omega^{\boldsymbol{\eta}}(t), t\in(0,T).$$

At the inlet and outlet boundaries, we prescribe zero tangential velocity and dynamic pressure:

$$\left.\begin{aligned} p + \frac{\rho_F}{2}|\mathbf{u}|^2 &= P_{in/out}(t), \\ \mathbf{u}\times\mathbf{e}_z &= 0, \end{aligned}\right\} \text{ on } \Gamma_{in/out}.$$

The shell

A clamped cylindrical Koiter shell of thickness h, length L, and reference radius of the middle surface R can be defined via parameterization

$$\boldsymbol{\varphi} : \omega \to \mathbb{R}^3, \quad \boldsymbol{\varphi}(z,\theta) = (z, R\cos\theta, R\sin\theta),$$

where $\omega = (0,L)\times(0,2\pi)$, and $R>0$. Under the action of force, the Koiter shell is displaced from its reference configuration Γ by a displacement $\boldsymbol{\eta} = \boldsymbol{\eta}(t,z,\theta) = (\eta_z, \eta_r, \eta_\theta)$.

Given a load $\mathbf{f}$, the displacement $\boldsymbol{\eta}$ of the Koiter shell is a solution to the following elastodynamics problem in weak form: find $\boldsymbol{\eta} = (\eta_z,\eta_r,\eta_\theta) \in V_K$ such that:

$$\rho_K h \int_\omega \partial_t^2 \boldsymbol{\eta}\cdot\boldsymbol{\psi} R + \langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\psi}\rangle = \int_\omega \mathbf{f}\cdot\boldsymbol{\psi} R, \ \forall \boldsymbol{\psi} \in V_K,$$

where V_K denotes the solution/test space equipped with the boundary conditions.

The mesh

We describe an elastic mesh as a three-dimensional elastic body defined as a union of three-dimensional struts. Since struts are slender or "thin", we approximate it with one-dimensional curved rod model. For the $i-$th curved rod, the middle line is parameterized via

$$\mathbf{P}_i : [0,l_i] \to \boldsymbol{\varphi}(\overline{\omega}), \quad i = 1,\dots,n_E,$$

and on each strut we have next family of equations:

$$\begin{aligned} \rho_S A_i \partial_t^2 \mathbf{d}_i &= \partial_s \mathbf{p}_i + \mathbf{f}_i, \\ \rho_S M_i \partial_t^2 \mathbf{w}_i &= \partial_s \mathbf{q}_i + \mathbf{t}_i \times \mathbf{p}_i, \\ 0 &= \partial_s \mathbf{w}_i - Q_i H_i^{-1} Q_i^T \mathbf{q}_i, \\ 0 &= \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i. \end{aligned}$$

Here, $\mathbf{d}_i$ is the displacement of the middle line of the $i-$th rod, $\mathbf{w}_i$ is the infinitesimal rotation of the cross-section of the $i-$th rod, $\mathbf{q}_i$ is the contact moment, and $\mathbf{p}_i$ is the contact force.

The fluid-composite structure coupling

In summary, we study the following fluid-structure interaction problem.

Problem 1. Find $(\mathbf{u}, p, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w})$ such that

$$\left.\begin{aligned} \rho_F\left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= \nabla\cdot\boldsymbol{\sigma} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}\right\} \text{ in } \Omega^{\boldsymbol{\eta}}(t), t\in(0,T),$$

$$\left.\begin{aligned} \partial_t\boldsymbol{\eta} &= (\mathbf{u}\circ\boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma \circ \boldsymbol{\varphi} \\ \rho_K h \partial_t^2 \boldsymbol{\eta} R + \mathcal{L}\boldsymbol{\eta} + \sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|}\delta_{J_i} &= -J((\boldsymbol{\sigma}\circ\boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma\circ\boldsymbol{\varphi})((\mathbf{n}\circ\boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma\circ\boldsymbol{\varphi}) \end{aligned}\right\} \text{ on } (0,T)\times\omega,$$

$$\left.\begin{aligned} \rho_S A_i \partial_t^2 \mathbf{d}_i &= \partial_s \mathbf{p}_i + \mathbf{f}_i \\ \rho_S M_i \partial_t^2 \mathbf{w}_i &= \partial_s \mathbf{q}_i + \mathbf{t}_i \times \mathbf{p}_i \\ 0 &= \partial_s \mathbf{w}_i - Q_i H_i^{-1} Q_i^T \mathbf{q}_i \\ 0 &= \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i \end{aligned}\right\} \text{ on } (0,T)\times(0,l_i), i\in(1,\dots,n_E).$$

Problem 1 is supplemented with the corresponding set of boundary and initial conditions. Furthermore, the formal energy estimates show that the total energy $E(t)$ of the problem is bounded by the data of the problem

$$\frac{d}{dt}E(t) + D(t) \le C(P_{in}(t), P_{out}(t)).$$

Definition of a weak solution

We say that $(\mathbf{u}, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in \mathcal{V}(0,T)$ is a weak solution of Problem 1, if for all test functions $(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in \mathcal{Q}(0,T)$ the following equality holds:

$$\begin{aligned} &\rho_F\left(-\int_0^T\int_{\Omega^{\boldsymbol{\eta}}(t)} \mathbf{u}\cdot\partial_t\boldsymbol{\upsilon} + \int_0^T b(t,\mathbf{u},\mathbf{u},\boldsymbol{\upsilon}) - \frac{1}{2}\int_0^T\int_{\Gamma^{\boldsymbol{\eta}}(t)} (\mathbf{u}\cdot\boldsymbol{\upsilon})(\mathbf{u}\cdot\mathbf{n})\right) \\ &+ 2\mu_F \int_0^T\int_{\Omega^{\boldsymbol{\eta}}(t)} \mathbf{D}(\mathbf{u}):\mathbf{D}(\boldsymbol{\upsilon}) - \rho_K h \int_0^T\int_\omega \partial_t\boldsymbol{\eta}\cdot\partial_t\boldsymbol{\psi} R + \int_0^T a_K(\boldsymbol{\eta},\boldsymbol{\psi}) \\ &- \rho_S \sum_{i=1}^{n_E} A_i \int_0^T\int_0^{l_i} \partial_t\mathbf{d}_i\cdot\partial_t\boldsymbol{\xi}_i - \rho_S\sum_{i=1}^{n_E}\int_0^T\int_0^{l_i} M_i\partial_t\mathbf{w}_i\cdot\partial_t\boldsymbol{\zeta}_i \\ &+ \int_0^T a_S(\mathbf{w},\boldsymbol{\zeta}) = \int_0^T \langle F(t),\boldsymbol{\upsilon}\rangle_{\Gamma_{in/out}} + \rho_F\int_\Omega \mathbf{u}_0\cdot\boldsymbol{\upsilon}(0) + \rho_K h\int_\omega \partial_t\boldsymbol{\eta}_0\cdot\boldsymbol{\psi}(0)R \\ &+ \rho_S\sum_{i=1}^{n_E} A_i\int_0^{l_i}\partial_t\mathbf{d}_{0i}\cdot\boldsymbol{\xi}_i(0) + \rho_S\sum_{i=1}^{n_E}\int_0^{l_i} M_i\partial_t\mathbf{w}_{0i}\cdot\boldsymbol{\zeta}_i(0). \end{aligned}$$

Assumption 1 *There exists a time $T>0$ such that for every $t \le T, \Gamma^{\boldsymbol{\eta}}(t)$ remains a subgraph of a function.*

Assumption 2 *There exists a constant $C>0$, independent of N, such that the structure displacements $(\boldsymbol{\eta}_N)_{N\in\mathbb{N}}$ satisfy: $\|\boldsymbol{\eta}_N\|_{C([0,T];W^{1,\infty}(\omega))} \le C$.*

The existence proof

- Define Arbitrary Lagrangian-Eulerian mapping to overcome the difficulties that arise due to the domain motion.
- Use the Lie operator splitting and semi-discretization to define a sequence of approximate solutions.
- Show the uniform boundedness of approximate solutions and extract weak converging subsequences.
- Use a version of Aubin-Lions-Simon lemma to show that sequences of approximate solutions are relatively compact in L^2 and extract strongly converging subsequences.
- Construct appropriate test functions and pass to the limit.

Main theorem

Let $\mathbf{u}_0 \in L^2(\Omega^{\boldsymbol{\eta}}(t))$, $\boldsymbol{\eta}_0 \in H^1(\omega)$, $\mathbf{v}_0 \in L^2(R;\omega)$, $(\mathbf{d}_0,\mathbf{w}_0) \in V_S$, $(\mathbf{k}_0,\mathbf{z}_0) \in L^2(\mathcal{N};\mathbb{R}^6)$ be such that

$$\nabla\cdot\mathbf{u}_0 = 0, \quad \mathbf{u}_0|_{\Gamma^{\eta_0}}\cdot\mathbf{n}^{\boldsymbol{\eta}_0} = \mathbf{v}_0\cdot\mathbf{n}^{\boldsymbol{\eta}_0}, \quad \boldsymbol{\eta}_0\circ\boldsymbol{\pi} = \mathbf{d}_0,$$

and let $P_{in/out} \in L^2_{loc}(0,\infty)$. Furthermore, let all the physical constants be positive: $\rho_K, \rho_S, \rho_F, \lambda, \mu, \mu_F > 0$ and $A_i > 0, \forall i = 1,\dots,n_E$. Assume that the uniform Lipschitz property specified in Assumption 2 holds, and that the subgraph property specified in Assumption 1 holds. Then, for every $t \le T$, where T is the maximal time for which the subgraph property holds, there exists a weak solution to Problem 1 satisfying the weak formulation.

References

1. Sunčica Čanić, Marija Galić and Boris Muha. Analysis of a 3d nonlinear, moving boundary problem describing fluid-mesh-shell interaction. *Trans. Amer. Math. Soc.*, **373** (2020), 9; 6621-6681.

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