Liouville type theorems in the stationary Navier-Stokes and related equations

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Abstract

We consider the stationary Navier-Stokes equations in $\mathbb{R}^3$

$$-\Delta u + (u \cdot \nabla)u = -\nabla p, \qquad \nabla \cdot u = 0. \tag{1}$$

The standard boundary condition to impose at the spatial infinity is

$$u(x) \to 0 \quad \text{as} \quad |x| \to 0. \tag{2}$$

We also assume finiteness of the Dirichlet integral,

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx < +\infty. \tag{3}$$

Obviously (u, p) with $u = 0$ and $p =$ constant is a trivial solution to (1)-(3). A very challenging open question is if there is another nontrivial solution. This Liouville type problem is wide open, and has been actively studied recently in the community of mathematical fluid mechanics. The explicit statement of the problem is written in Galdi's book[1, Remark X. 9.4, pp. 729], where under the stronger assumption $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$ he concludes $u = 0$. After that many authors deduce sufficient conditions stronger than (2) and/or (3) to obtain the Liouville type result. In this talk we review various previous results and present recent progresses in getting sufficient condition in terms of the potential functions of the velocity. We also show that similar method can applied to prove Liouville type theorems for the other related equations such as the magnetohydrodynamic equations(MHD), Hall-MHD and the non-Newtonian fluid equations. These are joint works with J. Wolf.

Suggested reading:

[1] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations: Steady-State Problems*, Springer, 2011.

[2] G. Seregin, *Remarks on Liouville type theorems for steady-state Navier-Stokes equations,* Algebra i Analiz **30**, (2018), no. 2, 238-248.

[3] G. Seregin and W. Wang, *Sufficient conditions on Liouville type theorems for the 3D steady Navier-Stokes equations,* Algebra i Analiz **31**. (2019), no. 2, 269-278.

[4] H. Kozono, Y. Terasawa, Y. Wakasugi, *A remark on Liouville-type theorems for the stationary Navier-Stokes equations in three space dimensions,* J. Func. Anal., **272**, (2017), 804-818.

[5] D. Chae and J. Wolf, *On Liouville type theorem for the stationary Navier-Stokes equations,* Calculus of Variations and PDEs, **58**, (2019), no.3, 58:111.

[6] D. Chae and J. Wolf, *On Liouville type theorems for the stationary MHD and Hall-MHD systems* J. Diff. Eqns., **295**, (2021), 233-243.

[7] D. Chae, *Liouville-type theorems for the forced Euler equations and the Navier-Stokes equations,* Comm. Math. Phys., **326**, (2014), pp. 37-48.

[8] D. Chae, *Note on the Liouville type problem for the stationary Navier-Stokes equations in* $\mathbb{R}^3$*,* J. Diff. Eqns, **268**, no. 3, (2020), 1043-1049.

[9] D. Chae, *Relative decay conditions on Liouville type theorem for the steady Navier-Stokes system.,* J. Math. Fluid Mech. **23**, (2021), no. 1, Paper No. 21, 6 pp.