Lecture 1: Water waves and Hamiltonian partial differential equations

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Outline

... the neverchanging everchanging water ... Ulysses – James Joyce

Free surface water waves

Symplectic forms and Hamiltonian PDEs

Hamiltonian PDE - examples

- Quasilinear wave equations
- Nonlinear Schrödinger equations
- Shallow water equations
- Boussinesq equations
- Korteweg deVries equations

A Hamiltonian for overturning waves

Euler's equations and free surface water waves

- Fluid domain $S(\eta) := \{x \in \mathbb{R}^{d-1},\ y \in (-h, \eta(x))\}$, $d = 2, 3$
- Incompressibility and irrotationality

$$\nabla \cdot \mathbf{u} = 0\ , \qquad \nabla \wedge \mathbf{u} = 0$$

therefore $\mathbf{u} = \nabla\varphi$ where

$$\Delta\varphi = 0$$

- On the solid bottom boundary of $S(\eta)$

$$N \cdot \mathbf{u} = 0$$

- On the free surface $\Gamma(\eta) := \{y = \eta(x)\}$

$$\begin{aligned} \partial_t \eta &= \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi \\ \partial_t \varphi &= -g\eta - \tfrac{1}{2}|\nabla\varphi|^2\ , \end{aligned}$$

Figure : Great waves off the Oregon coast

kinetic and potential energy

- The energy H of the system of equations is

$$\begin{aligned} H &= K + P := \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta(x)} \tfrac{1}{2}|\mathbf{u}|^2 \, dy dx + \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta(x)} gy \, dy dx \\ &= \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta(x)} \tfrac{1}{2}|\nabla\varphi|^2 \, dy dx + \int_{\mathbb{R}^{d-1}} \tfrac{g}{2}\eta^2 \, dx - C \, , \end{aligned}$$

- Rewrite the kinetic energy by integrating by parts

$$\begin{aligned} K &= \int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta(x)} \tfrac{1}{2}|\nabla\varphi|^2 \, dy dx = -\int_{\mathbb{R}^{d-1}} \int_{-h}^{\eta(x)} \tfrac{1}{2}\varphi\Delta\varphi \, dy dx \\ &\quad + \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\varphi N \cdot \nabla\varphi \, dS_{\text{bottom}} + \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\varphi N \cdot \nabla\varphi \, dS_{\text{free surface}} \end{aligned}$$

- Define $\xi(x) := \varphi(x, \eta(x))$, the kinetic energy is therefore

$$K = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\xi N \cdot \nabla\varphi \, dS_{\text{free surface}} = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\xi G(\eta)\xi \, dx$$

where $G(\eta)$ is the Dirichlet – Neumann operator.

Dirichlet – Neumann operator

- The Dirichlet – Neumann operator

$$G(\eta)\xi(x) = (\partial_y - \partial_x\eta(x)\cdot\partial_x)\varphi(x,\eta(x)) = R(N\cdot\nabla\varphi)(x,\eta(x))$$

where $R = \sqrt{1+|\partial_x\eta|^2}$ is a normalization factor so that $G(\eta)$ is self-adjoint on $L^2(dx)$.

- The Hamiltonian

$$H = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\xi G(\eta)\xi + \tfrac{g}{2}\eta^2\,dx$$

Theorem (Zakharov (1968))

The pair of functions $(\eta(x),\xi(x))$ are canonical variables for the water waves problem in which it can be written in Darboux coordinates, with Hamiltonian $H(\eta,\xi)$.

Hamiltonian formulation

- Therefore the equations for water waves can be rewritten in Darboux coordinates;

$$\begin{aligned} \dot{\eta} &= grad_{\xi} H = G(\eta)\xi \\ \dot{\xi} &= -grad_{\eta} H = -g\eta - grad_{\eta} K \end{aligned} \tag{1}$$

  The expressions for K and $grad_{\eta} K$ involve derivatives of $G(\eta)$ with respect to perturbations of the domain $S(\eta)$.

- Proposition

  Let $\eta \in C^1$. *Then* $G(\eta)$ *satisfies:*

  1. $G(\eta)$ *is positive semidefinite.*
  2. *It is self-adjoint (on an appropriately chosen domain).*
  3. $G(\eta)$ *maps* $H^1(\mathbb{R}^{d-1})$ *to* $L^2(\mathbb{R}^{d-1})$ *continuously.*
  4. *As an operator* $G(\eta) : H^1(\mathbb{R}^{d-1}) \to L^2(\mathbb{R}^{d-1})$ *it is analytic in* $\eta \in B_R(0) \subseteq C^1(\mathbb{R}^{d-1})$.

ZCS system

- Rewriting the system for water waves in these coordinates

$$\partial_t \eta = G(\eta)\xi$$
$$\partial_t \xi = -g\eta - grad_{\eta} K(\eta, \xi)$$

Variations of the kinetic energy with respect to the free surface $\eta(x)$ (the *shape derivative*) are given by

$$grad_{\eta} K(\eta, \xi) = \frac{1}{2(1 + |\partial_x \eta|^2)} \Big(|\partial_x \xi|^2 - (G(\eta)\xi)^2 - 2(\partial_x \eta \cdot \partial_x \xi) G(\eta)\xi + \big(|\partial_x \eta|^2 |\partial_x \xi|^2 - (\partial_x \eta \cdot \partial_x \xi)^2 \big) \Big)$$

- Variational formula of Hadamard (1910), (1916), related to the derivative of the Green's function with respect to the domain. Among other things, this is a useful formulation for numerical simulations

- Simulations using the Dirichlet – Neumann formulation

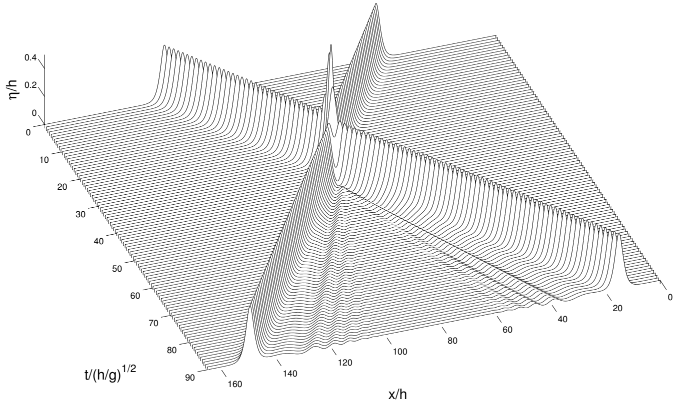


Figure : Head-on collision of two solitary waves, case $S/h = 0.4$
W. Craig, J. Hammack, D. Henderson, P. Guyenne & C. Sulem, Phys. Fluids 18, (2006)

proof by first principles of mechanics

- The Lagragian for surface water waves

$$L := K - P$$

  expressed in the tangent space variables $(\eta, \dot{\eta})$.
- Use the kinematic condition

$$\dot{\eta} = \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi = G(\eta)\xi$$

- The Lagrangian is thus

$$L(\eta, \dot{\eta}) = \frac{1}{2} \int_{\mathbb{R}^{d-1}} \dot{\eta} G^{-1}(\eta) \dot{\eta} - \frac{g}{2} \eta^2 \, dx$$

- The canonical conjugate variables are precisely those given by Zakharov.

$$(\eta, \partial_{\dot{\eta}} L) = (\eta, G^{-1}(\eta)\dot{\eta}) = (\eta, \xi)$$

Taylor expansion of $G(\eta)$

- The operator $G(\eta)\xi(x)$ is analytic, given by the expression
$$G(\eta)\xi = \sum_{j\geq 0} G^{(j)}(\eta)\xi$$
where each $G^{(j)}(\eta)$ is homogeneous of degree j in η
- Explicitely, for $D_x := -i\partial_x$
$$G^{(0)}\xi(x) = |D_x|\tanh(h|D_x|)\xi(x)$$
$$G^{(1)}(\eta)\xi(x) = D_x \cdot \eta D_x - G^{(0)}\eta G^{(0)}\xi(x)$$
$$G^{(2)}(\eta)\xi(x) = -\tfrac{1}{2}(G^{(0)}\eta^2 D_x^2 + D_x^2\eta^2 G^{(0)} - 2G^{(0)}\eta G^{(0)}\eta G^{(0)})\xi(x)$$

- Accordingly the Hamiltonian is analytic, with expansion
$$\begin{aligned} H(\eta,\xi) &= \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\xi G^{(0)}\xi + \tfrac{g}{2}\eta^2\,dx + \sum_{j\geq 3}\tfrac{1}{2}\int_{\mathbb{R}^{d-1}} \xi G^{(j-2)}(\eta)\xi\,dx \\ &= \sum_{j\geq 2} H^{(j)}(\eta,\xi) \end{aligned}$$

Hamiltonian PDE

- Phase space taken to be $v \in M$ a Hilbert space with inner product $\langle X, Y\rangle_M$, for $X, Y \in T(M)$
- Symplectic form given by a two form

$$\omega(X,Y) = \langle X, J^{-1}Y\rangle_M \ , \quad J^T = -J$$

  for tangent vectors $X, Y \in T(M)$. Antisymmetry $J^{-T} = -J^{-1}$
- Hamiltonian vector field X_H defined through the relation $dH(Y) = \omega(Y, X_H) = \langle \mathrm{grad}_v H, Y\rangle_M$ for all $Y \in T(M)$

$$\partial_t v = J \mathrm{grad}_v H(v) \ , \quad v(x,0) = v^0(x)$$

- The flow $v(x,t) = \varphi_t(v(x))$, defined for $v \in M_0 \subseteq M$
- Poisson brackets between H and other functions F are given by the expressions

$$\{H, F\}(v) := \frac{d}{dt}\Big|_{t=0} F(\varphi_t(v)) = \langle \mathrm{grad}_v H(v), J\mathrm{grad}_v F(v)\rangle_M$$

Poisson brackets and conservation laws

Express a conservation law using the Poisson bracket

$$\begin{aligned}\partial_t K(\eta(t,\cdot),\xi(t,\cdot)) &= \{H,K\} \\ &:= \int \mathrm{grad}_{\,\eta} K \,\mathrm{grad}_{\,\xi} H - \mathrm{grad}_{\,\xi} K \,\mathrm{grad}_{\,\eta} H \, dx\end{aligned}$$

- Conservation of mass $M(\eta) = \int \eta \, dx$

$$\begin{aligned}\{H,M\} &= \int \mathrm{grad}_{\,\eta} M \,\mathrm{grad}_{\,\xi} H - \mathrm{grad}_{\,\xi} M \,\mathrm{grad}_{\,\eta} H \, dx \\ &= \int 1\, G(\eta)\xi \, dx \\ &= \int G(\eta) 1\, \xi \, dx = 0\end{aligned}$$

- Momentum $I(\eta,\xi) = \int \eta \partial_x \xi \, dx$ $\partial_t I = \{H,I\} = 0$
- Energy $H(\eta,\xi)$ $\partial_t H = \{H,H\} = 0$

example 2: the quasilinear wave equation

- **Quasilinear wave equations**

$$\partial_t^2 u - \Delta u + \partial_t G_0(\partial_t u, \partial_x u) + \Sigma_j \partial_{x_j} G_j(\partial_t u, \partial_x u) = 0 \qquad (2)$$

  with initial data $u(0,x) = f(x)$ and $\partial_t u(0,x) = g(x)$

- This equation can be written as a Hamiltonian PDE. The Lagrangian is

$$L = \int \tfrac{1}{2}\dot{u}^2 - \tfrac{1}{2}|\partial_x u|^2 + G(\dot{u}, \partial_x u)\, dx$$

  Use the Legendre transform to change to canonical conjugate variables

$$p := \partial_{\dot{u}} L = p(\dot{u}, \partial_x u)$$

  The Hamiltonian is now

$$H(u,p) := \int \tfrac{1}{2}p^2 + \tfrac{1}{2}|\partial_x u|^2 + R(p, \partial_x u)\, dx$$

  where $R(p, \partial_x u) = -G(\dot{u}, \partial_x u)$ for $\dot{u} = \dot{u}(p, \partial_x u)$

quasilinear wave equation (cont.)

- Equation (2) can be rewritten as

$$\begin{aligned} \partial_t u &= \operatorname{grad}_p H(u,p) &&= p + \partial_p R \\ \partial_t p &= -\operatorname{grad}_u H(u,p) &&= \Delta u - \Sigma_j \partial_{x_j} \partial_{u_{x_j}} R(p, \partial_x u) \end{aligned}$$

which is in Darboux coordinates.

- Theorem (local existence theorem)

Systems of equations of this form are symmetrizable hyperbolic systems. Therefore for Sobolev data of sufficient smoothness $r > 0$

$$(u_0(x), p_0(x)) \in H^r(\mathbb{R}^d)$$

a solution exists locally in time

$$(u(t,x), p(t,x)) = \varphi_t(u_0, p_0) \in C([-T, +T] : H^r(\mathbb{R}^d)) := M_0 \subseteq L^2(\mathbb{R}^d)$$

Example 3: NLS

- **Nonlinear Schrödinger equation**
  On $\mathbb{R}^d$ or on a domain $\mathbb{T}^d = \mathbb{R}^d/\Gamma$, for period lattice Γ

$$i\partial_t u - \frac{1}{2}\Delta_x u + Q(x, u, \overline{u}) = 0 \qquad (3)$$

  with Hamiltonian

$$H_{NLS}(u) = \int \frac{1}{2}|\nabla u|^2 + G(x, u, \overline{u})\, dx\ , \quad \partial_{\overline{u}} G = Q$$

  Rewritten

$$\partial_t u = i\, \mathrm{grad}_{\overline{u}} H_{NLS}$$

- In many cases the Schrödinger equation admits a gauge symmetry under phase translation, in which case $G = G(x, |u|^2)$

example 4: Shallow water equations

- **Shallow water equations** Model equations for water waves

$$\partial_t \eta = -\partial_x \cdot \big((h+\eta)\partial_x \xi \big) \qquad (4)$$
$$\partial_t \xi = -g\eta - \tfrac{1}{2}|\partial_x \xi|^2$$

- This is a Hamiltonian PDE with Hamiltonian

$$H_{SW} = \frac{1}{2} \int (h + \eta(x))|\partial_x \xi|^2 + g\eta^2 \, dx$$

- The mathematically rigorous study of the shallow water limit of the equations of water waves was initiated by T. Kano & T. Nishida (1979).

  The work of D. Lannes (2000) extended their results from the analytic category to to Sobolev spaces, and to more subtle limiting situations such as the Green - Naghdi system

example 5: Boussinesq systems

- **Boussinesq system** on spatial domain $D \subseteq \mathbb{R}^1$
  The Hamiltonian is

$$H_{Boussinesq}(p,q) = \int_{\mathbb{T}^1} \frac{1}{2}p^2 + \frac{g}{2}q^2 \pm \frac{1}{12}(\partial_x p)^2 + G(x,q)\,dx$$

  The symplectic form is given by $J_{Boussinesq} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}$

$$\dot{p} = -\partial_x \mathrm{grad}_q H_{Boussinesq}(q,p) = -\partial_x(gq + \partial_q G) \tag{5}$$

$$\dot{q} = -\partial_x \mathrm{grad}_p H_{Boussinesq}(q,p) = -\partial_x(p \mp \frac{1}{6}\partial_x^2 p + \partial_p G)$$

- The $-$ sign is badly ill-posed (McKean, the *bad Boussinesq*), while the $+$ sign is well posed (the *good Boussinesq*).
  Completely integrable nonlinear cases include $G = p^3$ (Zakharov) and $G = \frac{1}{2}qp^2$ (Kaup, Sachs)

Example 6: the generalized KdV

- **Korteweg – de Vries equation**

$$\partial_t r = \frac{1}{6}\partial_x^3 r - \partial_x(\partial_r G(x,r)) \, , \quad x \in \mathbb{T}^1 \tag{6}$$

  The Hamiltonian is

$$H_{KdV}(r) = \int_{\mathbb{T}^1} \frac{1}{12}(\partial_x r)^2 + G(x,r)\, dx$$

  Rewritten

$$\partial_t r = J\, \mathrm{grad}_r H_{KdV} \, , \quad \text{where} \quad J = -\partial_x$$

  Completely integrable cases are $G = r^3$ and $G = r^4$

- Rigorous studies of water waves in the KdV scaling regime and limit are given in Craig (1985), T. Kano & T. Nishida (1986) and others

Water waves Hamiltonian in general coordinates

Following a question of T. Nishida in 2016

- Fluid domain $\Omega(t) \subseteq \mathbb{R}^2$ with free surface $\gamma(t,s)$.
  Evolution determined by free surface conditions

$$\mathbf{T}_{txy} \cdot \mathbf{N}_{txy} = 0 \,, \qquad p(t, \gamma(t,s)) = 0$$

- Energy = kinetic + potential

$$H = K + P \,, \qquad K = \frac{1}{2} \iint_{\Omega} |\nabla \varphi(x,y)|^2 \, dydx$$

  Potential energy

$$P = \iint_{\Omega} \nabla \cdot V \, dydx \,, \qquad V = (0, \frac{gy^2}{2})$$

- Dirichlet – Neumann operator, with $\varphi(\gamma(s)) = \xi(s)$

$$G(\gamma)\xi(s) := N \cdot \nabla \varphi(\gamma(s)) \,, \qquad K = \frac{1}{2} \int_{\gamma} \xi(s) G(\gamma) \xi(s) \, dS_{\gamma}$$

Legendre transform

- Lagrangian

$$L = K - P$$

The kinematic boundary condition states that

$$N \cdot \dot{\gamma} = N \cdot \nabla \varphi(\gamma) = G(\gamma)\xi$$

Decmpose the vector field $\dot{\gamma}(s)$ along the curve $\gamma = (\gamma_1(s), \gamma_2(s))$ in terms of its Frenet frame $(T(s), N(s))$

$$n(t,s) = N \cdot \dot{\gamma}(t,s) \ , \quad \tau(t,s) = T \cdot \dot{\gamma}(t,s)$$

then the Lagrangian is

$$L = \tfrac{1}{2} \int_\gamma n(t,s) G^{-1}(\gamma) n(t,s) \, dS_\gamma - \int_\gamma V \cdot N \, dS_\gamma$$

- The Legendre transform

$$\delta_{\dot{\gamma}} L = G^{-1}(\gamma) n(t,s) = \xi(s) \ ,$$

$$H = \tfrac{1}{2} \int_\gamma \xi(s) G(\gamma) \xi(s) \, dS_\gamma + \int_\gamma \frac{g}{2} \gamma_2^2(s) \frac{\partial_s \gamma_1(s)}{|\partial_s \gamma|} \, dS_\gamma$$

Hamilton's canonical equations

- variations $\delta\gamma$ and $\delta\xi$ of the Hamiltonian

$$N \cdot \dot{\gamma} = n = \delta_\xi K = G(\gamma)\xi$$

  the kinematic boundary conditions

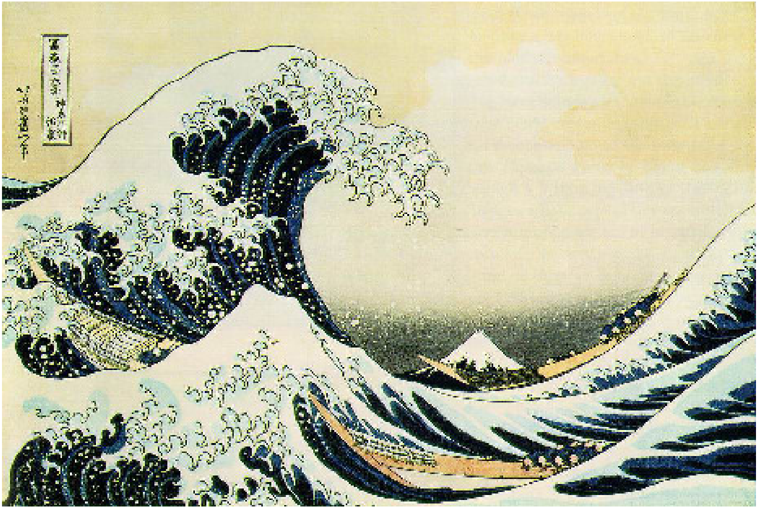
- Decomposing boundary variations $\delta\gamma = (N \cdot \delta\gamma)\, N + (T \cdot \delta\gamma)\, T$ variations of the potential energy

$$\delta_\gamma P \cdot \delta\gamma = \delta_{N \cdot \delta\gamma} P \cdot \delta\gamma = \int_\gamma g\gamma_2(s)\, N \cdot \delta\gamma\, dS_\gamma$$

- Finally $\delta_\gamma K$ has normal and tangential components, with the result that

$$\partial_t \xi = -g\gamma_2 - \tfrac{1}{2}\Big(\frac{1}{|\partial_s \gamma|^2} (\partial_s \xi)^2 - (G(\gamma)\xi)^2 - \frac{1}{|\partial_s \gamma|} \partial_s \xi \tau \Big)$$

  The tangential component $\tau = T \cdot \dot{\gamma}$ depends upon the manner in which the surface is parametrized, which in most cases imposes a holonomic constraint



Thank you