Viscous Liquid Flow Past an Obstacle at Arbitrary Reynolds Number

Giovanni P. Galdi

Department of Mechanical Engineering & Materials Science
and
Department of Mathematics

University of Pittsburgh

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A Little History and Some Basic Phenomenology

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Mathematical modeling and corresponding study of the *resistance of a liquid against a body* has a long history going back to the mid-700's.

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In 1748, likely motivated by the need of improving pipeline and ship design, the Berlin Academy proposed as the topic for the prize competition of 1750 the "theory of the resistance of fluids."

JEAN D'ALEMBERT, a winner of the previous Academy prize in 1746, submitted an essay to the Committee for their evaluation.

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- Introduction of the *velocity field* (vs. "velocity averages" of the Bernoullis). The velocity is allowed to vary from one place to another;

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- Introduction of the *velocity field* (vs. "velocity averages" of the BERNOULLIS). The velocity is allowed to vary from one place to another;
- Deduction of a set of equations that would nowadays be classified as those governing irrotational, plane flow of an incompressible fluid:

$$v_x = \frac{\partial \varphi}{\partial x}\,, \quad v_y = \frac{\partial \varphi}{\partial y}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi = 0$$

Despite its undoubted value –especially in retrospect– d'Alembert's manuscript was *not* awarded the Academy prize. The Committee decided that no manuscript submitted was good enough to earn the prize, by providing the official justification that "mathematical predictions were not compared with experiments," and postponed the competition to the following year 1751.

D'ALEMBERT became quite angry, because, in his view, such a requirement had not been made plain in the original statement of the problem. As a result, he at once decided to withdraw his manuscript, which he published in 1752 in an enlarged book form (*"Essai d'une Nouvelle Théorie de la Résistance des Fluides"*), where, among other things, he extended his theory to include the more general case of axially–symmetric flow.

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For the record, the prize was eventually awarded in 1752 to JACOBO ADAMI, an apparently amateur Italian mathematician.

The president of the prize Committee was LEONHARD EULER. There is little doubt that EULER picked up D'ALEMBERT painful pioneering efforts and, eventually, expanded them into a series of three fundamental papers published in 1757 in the *Memoires de l'Academie Royale des Sciences de Berlin*.

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$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0.$$

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right)$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$

$$R - \frac{1}{q}\left(\frac{dp}{dz}\right) = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right)$$

d'Alembert's "revenge" against Euler came a little later, in 1768, when he showed that Euler's model is unable to give any explanation of the force, $\boldsymbol{F}$, exerted on an obstacle, $\mathcal{B}$, fully submerged in the stream of a liquid (*d'Alembert Paradox*).

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A modern version of the paradox goes as follows.

The motion is assumed steady and *irrotational*:

$$\boldsymbol{v} = \nabla\varphi\,,\ \ \Delta\varphi = 0 \ \text{ in } \Omega = \mathbb{R}^3 - \mathcal{B}\,;$$

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0\,, \quad \lim_{|x|\to\infty} \nabla\varphi = \boldsymbol{U}$$

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$$p = -\tfrac{1}{2}\rho\left[(\nabla\varphi)^2 - \boldsymbol{U}\cdot\boldsymbol{U}\right].$$

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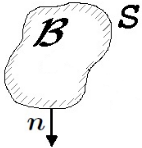

force of the liqid on $\mathcal{B}$

$$\boldsymbol{F} := -\int_S p\,\boldsymbol{n}$$

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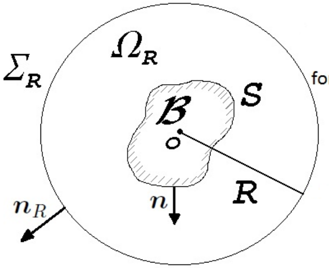

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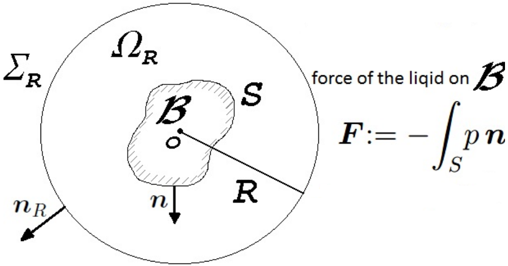
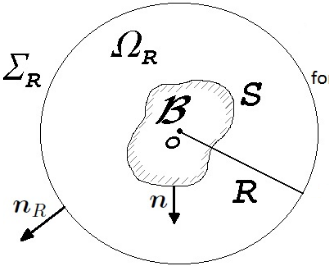


Integrating Euler equations on Ω_R:

$$\boldsymbol{F} := -\int_S p\,\boldsymbol{n} = -\tfrac{1}{2}\rho \int_{\Sigma_R} (\nabla\varphi - \boldsymbol{U}) \cdot (\nabla\varphi + \boldsymbol{U})\,\boldsymbol{n}_R$$

$$-\int_{\Sigma_R} \left[(\nabla\varphi - \boldsymbol{U}) \cdot \boldsymbol{n}_R \nabla\varphi + \boldsymbol{U} \cdot \boldsymbol{n}_R (\nabla\varphi - \boldsymbol{U}) \right]$$



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$$-\int_{\Sigma_R}\left[(\nabla\varphi - \boldsymbol{U})\cdot\boldsymbol{n}_R\nabla\varphi + \boldsymbol{U}\cdot\boldsymbol{n}_R(\nabla\varphi - \boldsymbol{U})\right]$$

$$\nabla\varphi - \boldsymbol{U} = O(R^{-3}) \implies \boldsymbol{F} = \boldsymbol{0}$$

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Heuristic: At very large distances from $\mathcal{B}$, the flow is uniform $(\boldsymbol{v} = \boldsymbol{U})$, so that the vorticity $\boldsymbol{\omega}(\boldsymbol{X}, t)$ of the particle $\boldsymbol{X}$ at time t must satisfy (say) $\boldsymbol{\omega}(\boldsymbol{X}, 0) = \mathbf{0}$. By Helmholtz theorem,

$$\boldsymbol{\omega}(\boldsymbol{X}, t) = \boldsymbol{F} \cdot \boldsymbol{\omega}(\boldsymbol{X}, 0)\,,$$

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Thus $\boldsymbol{\omega}(\boldsymbol{X}, t) \equiv \mathbf{0}$, provided the inverse map $\boldsymbol{x} \to \boldsymbol{X}$ exists and is smooth enough.

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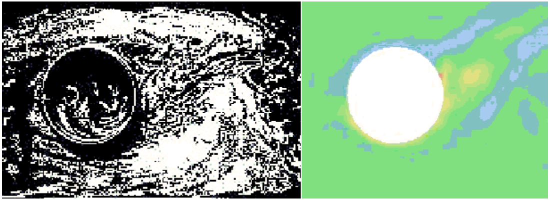
Recently, Hoffman & Johnson (JMFM 2010) have provided numerical evidence that irrotational flows are unstable to small perturbations. Instead, they found different solutions showing substantial *nonzero* drag and lift



As is known, d'Alembert paradox can be resolved by using the model introduced by Navier (1822), which takes into account the *viscosity* of the liquid:

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$$P - \frac{dp}{dx} = \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right) - \varepsilon\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right),$$

$$Q - \frac{dp}{dy} = \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) - \varepsilon\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right),$$

$$R - \frac{dp}{dz} = \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right) - \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right).$$

A.C., Bull. Sci. Math. Phys. Chim., **5**, (1828) p. 13

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> *Besides, Mr. Navier himself states that his basic principle is merely a hypothesis that only experience can verify.*
> *But if ordinary formulas of hydrodynamics are already so rebellious to the analysis, what should we expect from new formulas that are much more complicated?*

Stokes formula for the drag (1851)

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Setting

$$\boldsymbol{T}(\boldsymbol{v},p) = \rho\,\nu\big[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^{\top}\big] - p\boldsymbol{I}\,,$$

the *drag*, F_D, on the body $\mathcal{B}$ is defined as

$$F_D = \boldsymbol{U}\cdot\int_{\partial\mathcal{B}} \boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n}\,; \quad \boldsymbol{n} = \text{outer normal to } \partial\mathcal{B}\,.$$

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- $\mathcal{B}$ is a sphere of radius R;
- motion is steady and "slow" (nonlinearity neglected);

$$F_D = 6\,\pi\,\rho\,\nu\,R\,U\,.$$

Oseen Refinement of Stokes Formula (1927)

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By replacing the nonlinear term in the Navier-Stokes equations with

$$\boldsymbol{U} \cdot \nabla \boldsymbol{v} \,,$$

Oseen provided the following, more accurate approximation of F_D:

$$F_D = 6\,\pi\,\rho\,\nu\,R\,U\Big(1 + \tfrac{3}{8}\mathrm{Re} + O(\mathrm{Re}^2)\Big)\,,$$

$$\mathrm{Re} := \frac{U\,R}{\nu}\,.$$

A further refinement of Oseen formula, was later achieved by Proudman and Pearson (JFM, 1957) through a semi-quantitative argument (*matching asymptotics expansion*), and on an entirely rigorous ground by using the *fully nonlinear* equations, by Fischer, Hsiao & Wendland (JMAA, 1985):

$$F_D = 6\,\pi\,\rho\,\nu\,R\,U\left(1+\tfrac{3}{8}\mathrm{Re}+\tfrac{9}{40}\mathrm{Re}^2\ln\mathrm{Re}+O(\mathrm{Re}^2)\right).$$

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That's how far mathematical analysis goes.

How do these results compare with the *real* world?

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Introduce the dimensionless *drag coefficient* C_D:

$$F_D = \tfrac{\pi}{4}\,\rho\,C_D\,U^2\,R^2\,.$$

By a simple dimensional analysis one shows that

$$C_D = C_D(\mathrm{Re})\,,\quad \mathrm{Re} = \frac{U\,R}{\nu}$$

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For very slow (Stokes) flow, $F_D = 6\pi\rho\nu\, U R$, and

$$C_D = \frac{24}{\mathrm{Re}}\,.$$

Experimental Finding for C_D in Flow Past a Sphere

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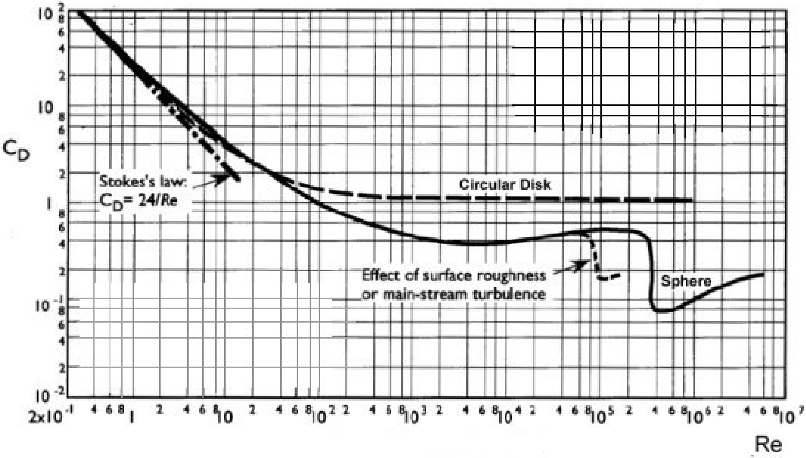
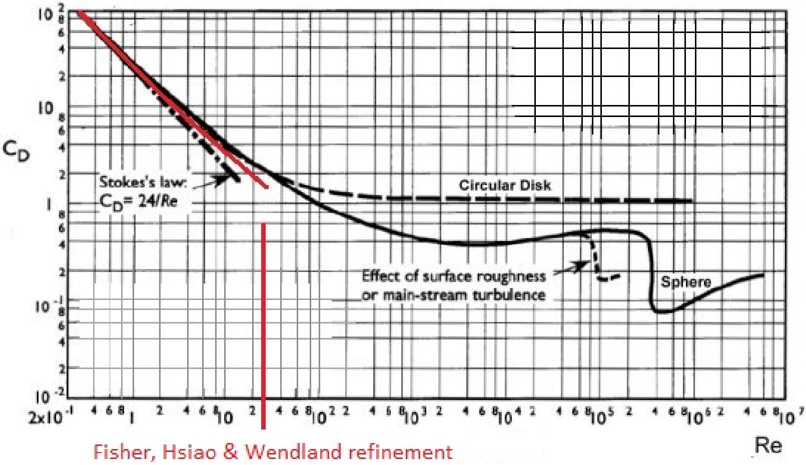


Experimental Finding for C_D in Flow Past a Sphere



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What is the reason for the discrepancy between the observed and the predicted values, even at "small" Reynolds numbers?

- The drag is calculated along *steady* and *laminar* flow (i.e. corresponding solutions exist for *all* Reynolds numbers) ;
- As the Reynolds number is more and more increased, the motion of the liquid is neither laminar nor steady. Actually, the dynamics is very complex, even before turbulence sets in.

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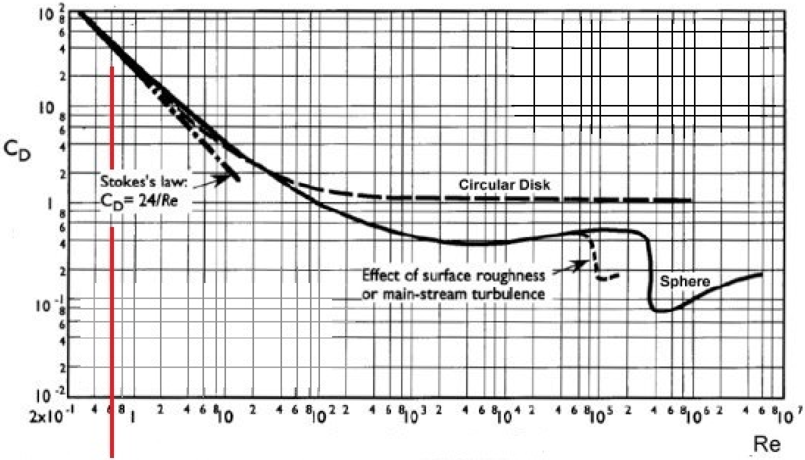
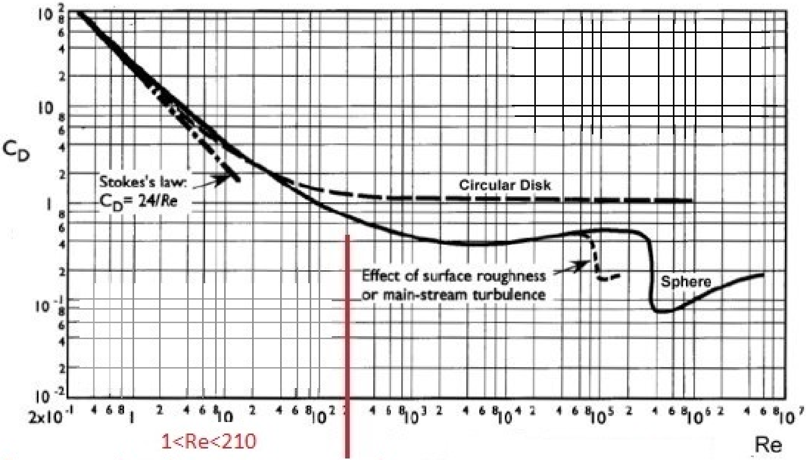


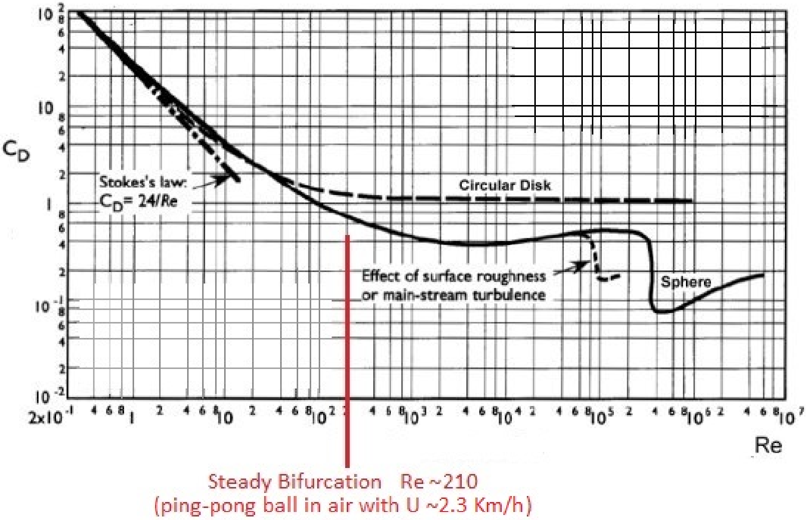
flow is steady and reversible: 0<Re<1
(ping-pong ball in air with U~11.5 m/h)

Experimental Finding for C_D in Flow Past a Sphere



flow is steady with an axisymmetric wake
(ping-pong ball with 11.5 m/h<U<2.3 Km/h)

Experimental Finding for C_D in Flow Past a Sphere



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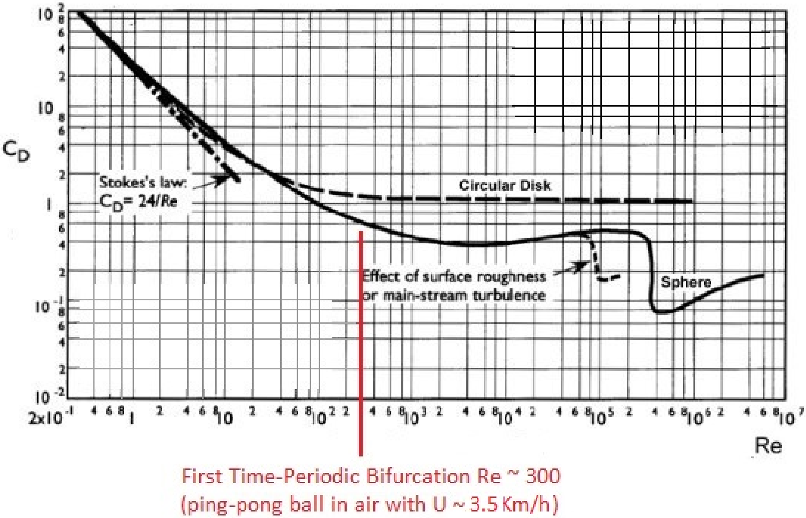
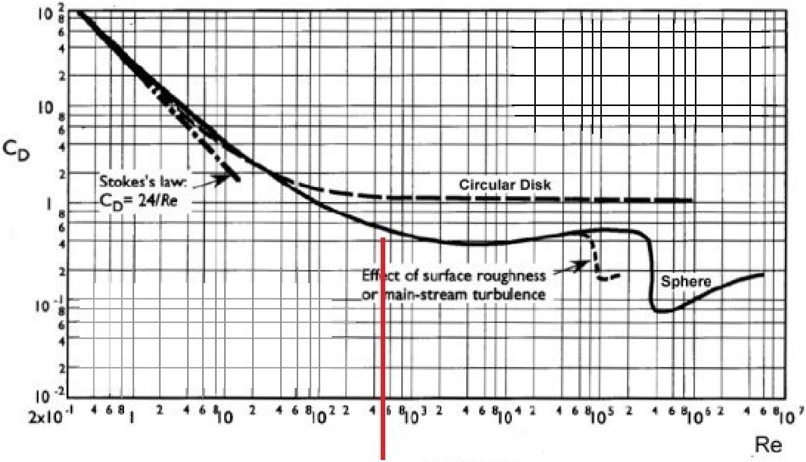
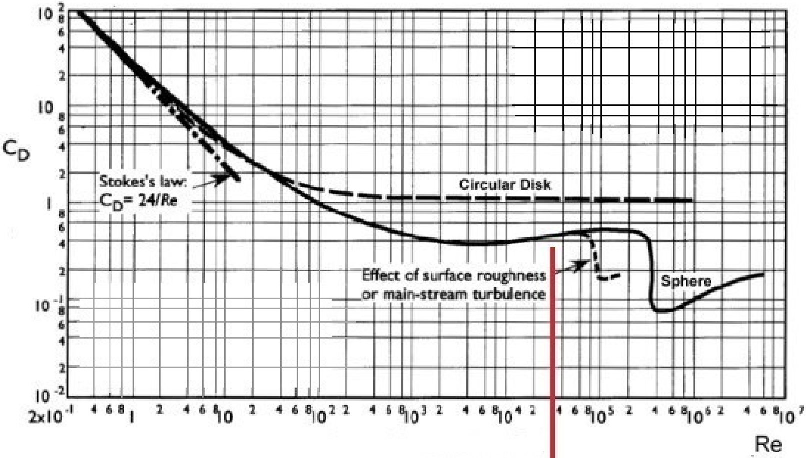


Experimental Finding for C_D in Flow Past a Sphere



Second Time-Periodic Bifurcation Re ~ 500
(ping-pong ball in air with U ~5.7 Km/h)

Experimental Finding for C_D in Flow Past a Sphere



Fully Turbulent
(ping-pong ball in air with U ~ 115 Km/h)

At Reynolds number around 10^5 the drag coefficient drops dramatically:

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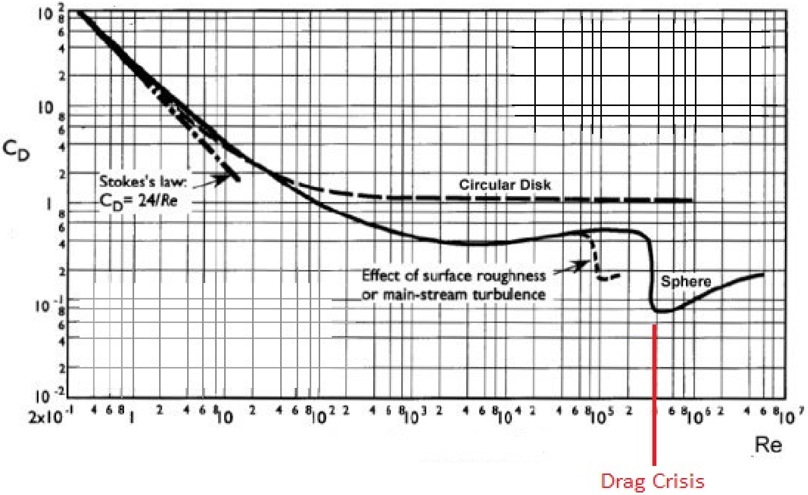
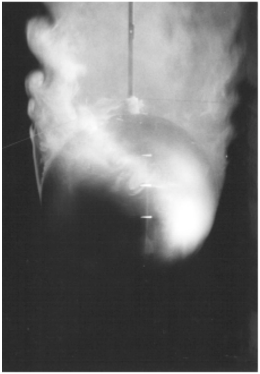
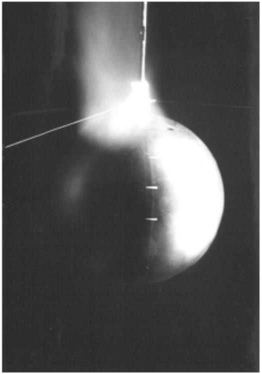


This phenomenon is due to a sudden size reduction of the wake behind the body





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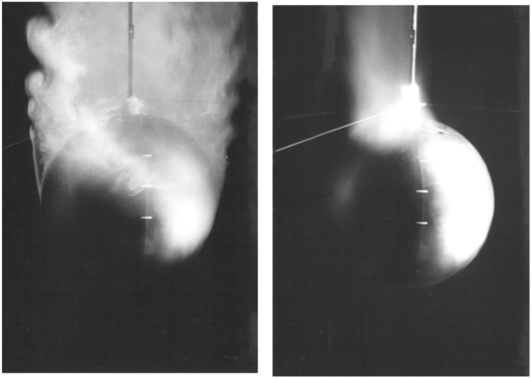
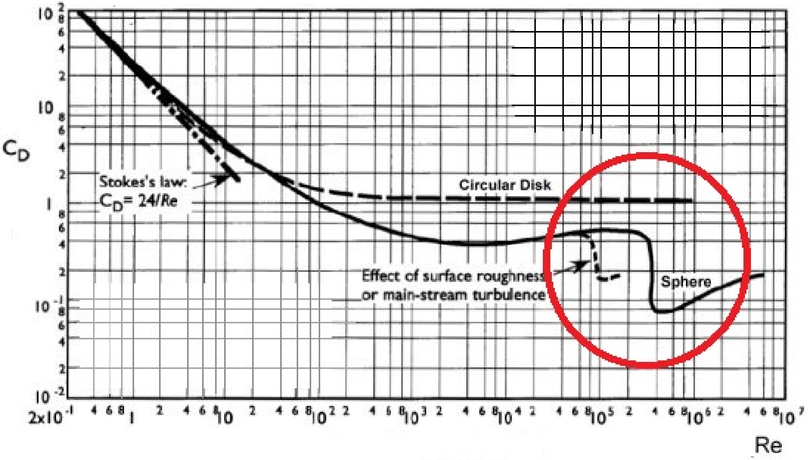


A ball thrown in the air that reaches those Reynolds numbers can keep its speed for longer

Drag crisis occurs at *lower* Reynolds number if the surface is *rough*:



Application to Golf:

Application to soccer:

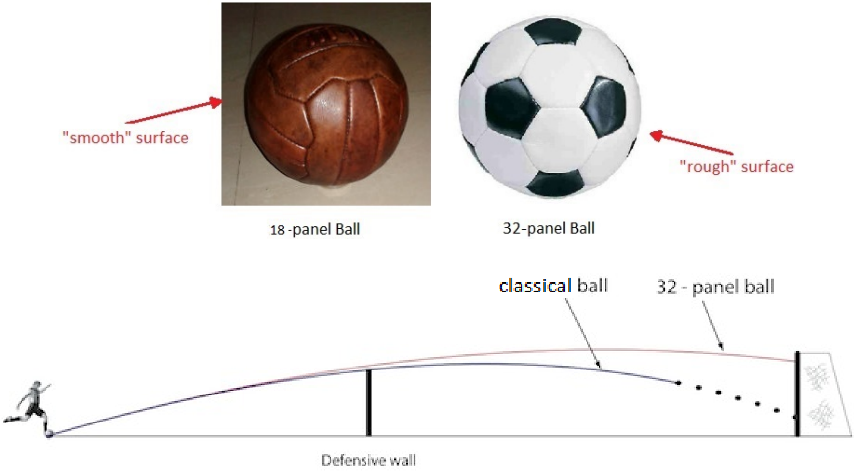
Application to soccer:

18 -panel Ball

32-panel Ball

Application to soccer:

Application to soccer:



Summary and Qualitative Bifurcation Diagram

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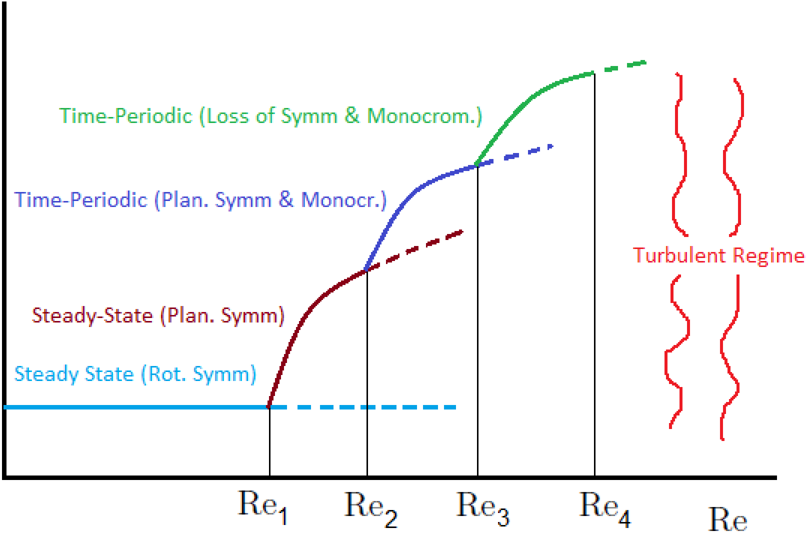

Mathematical Modeling

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We assume that the (Navier-Stokes) liquid fills the whole space, Ω, outside a body $\mathcal{B}$, driven by a uniform flow, of constant velocity $\boldsymbol{U}$, at large distances from $\mathcal{B}$. Let $\boldsymbol{U} = U\,\boldsymbol{e}_1$, $d = \operatorname{diam}(\mathcal{B})$.

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Relevant flow equations in dimensionless form ($\partial_t \equiv \partial/\partial t$):

$$\left.\begin{array}{c}\partial_t \boldsymbol{v} + \lambda\,\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \Delta\boldsymbol{v} - \nabla p\\ \mathrm{div}\,\boldsymbol{v} = 0\end{array}\right\}\ \text{in } \Omega\times(0,\infty)$$

$$\boldsymbol{v}(x,t) = \boldsymbol{0}\,,\quad \boldsymbol{x}\in\partial\Omega\,,\quad \lim_{|x|\to\infty}\boldsymbol{v}(x,t) = \boldsymbol{e}_1\,,\quad t\geq 0$$

$$\lambda \equiv \mathrm{Re} = \frac{U d}{\nu}\,.$$

Notation.

$$\mathcal{D}_0^{1,2}(\Omega) :=$$

$$\left\{ \boldsymbol{u} \in L^1_{\text{loc}}(\Omega) : \ \nabla \boldsymbol{u} \in L^2(\Omega)\,, \ \operatorname{div} \boldsymbol{u} = 0\,, \ \boldsymbol{u}|_{\partial\Omega} = \mathbf{0} \right\}$$

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Solenoidal Extension(Leray-Hopf)

Let $\boldsymbol{V} = \boldsymbol{V}(\lambda) \in C_0^{\infty}(\overline{\Omega})$ such that

(a) $\boldsymbol{V}|_{\partial\Omega} = -\boldsymbol{e}_1$;

(b) $\operatorname{div} \boldsymbol{V} = 0$;

(c) $-\int_{\Omega} \boldsymbol{u} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u} \leq \frac{1}{2} \lambda \|\nabla \boldsymbol{u}\|_2^2$, all $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$.

Setting $\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{V} - \boldsymbol{e}_1$, the relevant problem becomes ($\partial_1 \equiv \partial/\partial x_1$)

$$\left.\begin{aligned} &\partial_t \boldsymbol{u} + \lambda(\partial_1 \boldsymbol{u} + \boldsymbol{V}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{V}) \\ &\qquad = -\lambda\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \Delta\boldsymbol{u} - \nabla p + \boldsymbol{H} \\ &\operatorname{div}\boldsymbol{u} = 0 \end{aligned}\right\}$$

$$\boldsymbol{u}(x,t) = \boldsymbol{0}\,, \quad \boldsymbol{x} \in \partial\Omega\,, \qquad \lim_{|x|\to\infty} \boldsymbol{u}(x,t) = \boldsymbol{0}$$

with

$$\boldsymbol{H} := -\lambda(\boldsymbol{e}_1 + \boldsymbol{V})\cdot\nabla\boldsymbol{V} + \Delta\boldsymbol{V}$$

Steady-State Solutions

Steady-state solutions ($\partial \boldsymbol{u}/\partial t \equiv \mathbf{0}$) must then satisfy :

$$\left.\begin{aligned} &\Delta \boldsymbol{u} - \lambda(\partial_1 \boldsymbol{u} + \boldsymbol{V}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{V}) - \nabla p \\ &\qquad = \lambda \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \boldsymbol{H} \\ &\operatorname{div}\boldsymbol{u} = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\boldsymbol{u}(x) = \mathbf{0}\,, \quad \boldsymbol{x} \in \partial\Omega\,, \quad \lim_{|x|\to\infty} \boldsymbol{u}(x) = \mathbf{0}$$

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$$\boldsymbol{u}(x) = \mathbf{0}\,, \quad \boldsymbol{x} \in \partial\Omega\,, \quad \lim_{|x|\to\infty} \boldsymbol{u}(x) = \mathbf{0}$$

To study the properties of these solutions for arbitrary $\lambda \in (0,\infty)$ it is convenient to reformulate the problem in an appropriate Banach space.

Steady-State Solutions: Generic Properties and Bifurcation

Let $\mathcal{D}_0^{-1,2}(\Omega) = \big(\mathcal{D}_0^{1,2}(\Omega)\big)'$, and define

$$X(\Omega) = \big\{\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega) : \ \partial_1\boldsymbol{u} \in \mathcal{D}_0^{-1,2}(\Omega)\big\}$$

where

$$\partial_1\boldsymbol{u} \in \mathcal{D}_0^{-1,2}(\Omega) \Leftrightarrow \sup_{\boldsymbol{\varphi}\in\mathcal{D}_0^{1,2}(\Omega)} \frac{|\int_\Omega \partial_1\boldsymbol{u}\cdot\boldsymbol{\varphi}|}{\|\nabla\boldsymbol{\varphi}\|_2} := |\partial_1\boldsymbol{u}|_{-1,2} < \infty.$$

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$X(\Omega)$ is a separable, reflexive Banach space when endowed with the "natural" norm:

$$\|\boldsymbol{u}\|_{X(\Omega)} := \|\nabla \boldsymbol{u}\|_2 + |\partial_1 \boldsymbol{u}|_{-1,2} \, .$$

Two fundamental properties (GPG '07)

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Lemma 1. $X(\Omega) \subset L^4(\Omega)$ and

$$\|\boldsymbol{u}\|_4 \leq C\, |\partial_1 \boldsymbol{u}|_{-1,2}^{\frac{1}{4}} \|\nabla \boldsymbol{u}\|_2^{\frac{3}{4}}\,.$$

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Corollary

$$\boldsymbol{u} \in X(\Omega) \implies \boldsymbol{u} \cdot \nabla \boldsymbol{u} \equiv \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) \in \mathcal{D}_0^{-1,2}(\Omega)$$

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Remark. If *merely* $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ we can *only* deduce (Sobolev) $\boldsymbol{u} \in L^6(\Omega)$. Therefore $X(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega)$ is a "more regular" space at infinity.

Define the (linear) *"Oseen Operator"*

$$\mathcal{L} : (\lambda, \boldsymbol{u}) \in (0, \infty) \times X(\Omega) \mapsto$$
$$\Delta \boldsymbol{u} - \lambda(\partial_1 \boldsymbol{u} + \boldsymbol{V} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{V}) \in \mathcal{D}_0^{-1,2}(\Omega)$$

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(well-defined because of the Corollary) ;
and

$$\mathcal{H} : \lambda \in (0, \infty) \mapsto \lambda\, (\boldsymbol{e}_1 + \boldsymbol{V}) \cdot \nabla \boldsymbol{V} - \Delta \boldsymbol{V} \in \mathcal{D}_0^{-1,2}(\Omega)$$

The steady-state problem can be then reformulated as the equation:

$$\mathcal{M}(\lambda, \boldsymbol{u}) := \mathcal{L}(\lambda, \boldsymbol{u}) + \mathcal{H}(\lambda) + \mathcal{N}(\lambda, \boldsymbol{u}) = 0 \text{ in } \mathcal{D}_0^{-1,2}(\Omega)\,.$$

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We would like to investigate generic properties of the *solution manifold*:

$$\mathfrak{M} := \big\{(\lambda,\boldsymbol{u}) \in (0,\infty)\times X(\Omega) :\ \mathcal{M}(\lambda,\boldsymbol{u}) = 0\big\}$$

and associated *level set*:

$$\mathfrak{S}(\lambda_0) = \big\{\boldsymbol{u} \in X(\Omega) :\ \mathcal{M}(\lambda_0,\boldsymbol{u}) = 0\big\}$$

Steady-State Solutions: Generic Properties and Bifurcation

(i) Is $\mathfrak{S}(\lambda_0) \neq \emptyset$? (Existence) ;

(ii) When is $\dim(\mathfrak{S}(\lambda_0)) = 1$? (Global Uniqueness)

(iii) What is $\dim(\mathfrak{S}(\lambda_0))$, in general? (How many solutions for a given λ_0)

(iv) What is the geometric structure of $\mathfrak{M}$?

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Given a *solution branch* $(\lambda, \boldsymbol{u}) \in \mathfrak{M}$, $\lambda \in U(\lambda_0)$, the point $(\lambda_0, \boldsymbol{u}_0)$ is a *steady bifurcation point* if there is $(\lambda_n, \boldsymbol{w}_n) \in \mathfrak{M}$ such that

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(v) Ho do we characterize steady bifurcation points?

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A *linear* operator $L : X \to Y$ is Fredholm of index $m \in \mathbb{N}$ if (N = null space; R = range)

$$\alpha := \dim \mathsf{N}(L) < \infty\,, \quad \beta := \operatorname{codim} \mathsf{R}(L) < \infty\,;$$

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A *nonlinear* map $M \in C^1(X, Y)$ with $\mathsf{D}(M) = X$ is Fredholm of index $m \in \mathbb{N}$, if $M'(x)$ is Fredholm of index m for all $x \in X$.

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A map $M : X \to Y$ is *proper* if $M^{-1}(C)$ is compact in X for every compact $C \subset Y$.

Theorem 1. (GPG '07) $M \in C^2(X, Y)$ is a proper Fredholm map of index 0 satisfying the following properties.

(i) There exists $\overline{y} \in Y$ such that $M(x) = \overline{y}$ has one and only one solution $\overline{x}$;

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Then the following properties hold.

(a) For any $y \in Y$, $M(x) = y$ has one solution;

(b) There exists an open, dense (residual) set $Y_0 \subset Y$ such that for any $y \in Y_0$ ("almost all" $y \in Y$) the equation $M(x) = y$ has an odd number, $\kappa = \kappa(y)$, of solutions.

Apply the theorem to $M \equiv \mathcal{M}(\lambda, \cdot)$, for a *fixed* $\lambda > 0$. Give for granted, momentarily, Fredholm property and properness and check (i) and (ii).

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$$\begin{aligned} &\Delta \boldsymbol{u} - \lambda(\partial_1 \boldsymbol{u} + \boldsymbol{V} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{V} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \nabla p \\ &\operatorname{div} \boldsymbol{u} = 0 \end{aligned}$$

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$$\Longrightarrow \; \|\nabla \boldsymbol{u}\|_2^2 = -\lambda \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u} \leq \tfrac{1}{2} \|\nabla \boldsymbol{u}\|_2^2 \;\; \Longrightarrow \;\; \boldsymbol{u} = \boldsymbol{0}$$

Assumption (ii) then reduces to show that the *linearization around* $\boldsymbol{u} = \mathbf{0}$:

$$\begin{aligned} &\Delta \boldsymbol{w} - \lambda(\partial_1 \boldsymbol{w} + \boldsymbol{V} \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{V}) = \nabla p \\ &\operatorname{div} \boldsymbol{w} = 0 \end{aligned}$$

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As before,

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For fixed $\lambda > 0$ the derivative (linearization), $\mathcal{M}_{\boldsymbol{u}}$, of $\mathcal{M}(\lambda, \cdot)$ at $\boldsymbol{u} \in X(\Omega)$ is:

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We have to show that there is $\{\boldsymbol{w}_{n'}\} \subseteq \{\boldsymbol{w}_n\}$ such that, as $n' \to \infty$,

$$|\boldsymbol{u} \cdot \nabla \boldsymbol{w}_{n'}|_{-1,2} := \sup_{\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)} \frac{|(\boldsymbol{u} \cdot \nabla \boldsymbol{w}_{n'}, \boldsymbol{\varphi})|}{\|\nabla \boldsymbol{\varphi}\|_2} \to 0 ,$$

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$$(\boldsymbol{u}_1, \boldsymbol{u}_2) := \int_\Omega \boldsymbol{u}_1 \cdot \boldsymbol{u}_2 .$$

Since $\|\boldsymbol{w}_n\|_{X(\Omega)} = 1$, by embedding (Lemma 1)

$$\|\boldsymbol{w}_n\|_4 + \|\nabla \boldsymbol{w}_n\|_2 \leq C\,.$$

Steady-State Solutions: Generic Properties and Bifurcation

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Thus, uniformly in $\boldsymbol{\varphi} \in \mathcal{D}^{1,2}_0(\Omega)$, as $n', R \to \infty$:

$$\begin{aligned}
&|(\boldsymbol{u}\cdot\nabla \boldsymbol{w}_{n'}, \boldsymbol{\varphi})| \\
&\quad = |(\text{div}\,(\boldsymbol{u} \otimes \boldsymbol{w}_{n'}), \boldsymbol{\varphi})| = |(\boldsymbol{u} \otimes \boldsymbol{w}_{n'}, \nabla \boldsymbol{\varphi})| \\
&\quad \leq \left(\|\boldsymbol{u}\|_4 \|\boldsymbol{w}_{n'}\|_{4,\Omega_R} + \|\boldsymbol{u}\|_{4,\Omega^R} \|\boldsymbol{w}_{n'}\|_4\right) \|\nabla \boldsymbol{\varphi}\|_2 \\
&\qquad \to 0
\end{aligned}$$

Properness.

Properness. Classical Leray-Schauder: $M : X \mapsto Y$

(A) $M = H + N$, H homeomorphism, N compact ;

(B) There is $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ mapping bounded set into bounded set such that

$$\|x\|_X \leq \phi(\|M(x)\|_Y) \quad \text{(a priori estimate)} .$$

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In our case (fixed λ):

$$\mathcal{M} = \underbrace{\Delta \boldsymbol{u} - \lambda(\partial_1 \boldsymbol{u} + \boldsymbol{V} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{V})}_{\text{Homeomorphism}}$$

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Properness. Lemma 2 (GPG '14) Suppose

(A) $M = H + N$, H homeomorphism, N quadratic;

(B) $N'(x)$ compact at every $x \in X$;

(C) There is $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ mapping bounded set into bounded set, with $\psi(s) \to 0$ as $s \to 0$, such that

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Then M is proper.

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Take $M \equiv \mathcal{M}(\lambda, \cdot) - \lambda \boldsymbol{V} \cdot \nabla \boldsymbol{V} + \Delta \boldsymbol{V}$.

We already checked conditions (A) and (B)

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Condition (C) means that given $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$ all corresponding solution $\boldsymbol{u} \in X(\Omega)$ to

$$\left.\begin{aligned} &\Delta \boldsymbol{u} - \lambda(\partial_1 \boldsymbol{u} + \boldsymbol{V} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{V}) \\ &\qquad\qquad = \lambda\, \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p + \boldsymbol{f} \\ &\operatorname{div} \boldsymbol{u} = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\boldsymbol{u} = \boldsymbol{0} \ \text{ at } \partial\Omega$$

satisfy

$$\|\nabla \boldsymbol{u}\|_2 + |\partial_1 \boldsymbol{u}|_{-1,2} \leq \psi(|\boldsymbol{f}|_{-1,2})$$

<u>1st Estimate.</u> (Classical, due to Leray)

$$
\begin{aligned}
\|\nabla \boldsymbol{u}\|_2^2 &= -\lambda(\boldsymbol{u}\cdot\nabla \boldsymbol{V}, \boldsymbol{u}) - \langle \boldsymbol{f}, \boldsymbol{u}\rangle \\
&\Longrightarrow \|\nabla \boldsymbol{u}\|_2^2 \le \tfrac{1}{2}\|\nabla \boldsymbol{u}\|_2^2 + |\boldsymbol{f}|_{-1,2}\|\nabla \boldsymbol{u}\|_2 \\
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2nd Estimate. Recall that the *Oseen operator*

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$$

is a homeomorphism:

$$
\|\boldsymbol{u}\|_{X(\Omega)} \equiv \|\nabla \boldsymbol{u}\|_2 + |\partial_1 \boldsymbol{u}|_{-1,2} \le C\,|\mathfrak{L}(\boldsymbol{u})|_{-1,2}\,.
$$

Thus, from

$$\left.\begin{aligned} &\Delta \boldsymbol{u} - \lambda(\partial_1 \boldsymbol{u} + \boldsymbol{V} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{V}) \\ &\qquad\qquad\qquad = \lambda\, \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p + \boldsymbol{f} \\ &\operatorname{div} \boldsymbol{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{at } \partial\Omega$$

we get

$$\begin{aligned} \|\nabla \boldsymbol{u}\|_2 + |\partial_1 \boldsymbol{u}|_{-1,2} &\le C \left(|\boldsymbol{u} \cdot \nabla \boldsymbol{u}|_{-1,2} + |\boldsymbol{f}|_{-1,2}\right) \\ &\le C \left(\|\boldsymbol{u}\|_4^2 + |\boldsymbol{f}|_{-1,2}\right) \end{aligned}$$

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$$\begin{aligned} \|\nabla \boldsymbol{u}\|_2 + |\partial_1 \boldsymbol{u}|_{-1,2} &\le C\left(|\boldsymbol{u} \cdot \nabla \boldsymbol{u}|_{-1,2} + |\boldsymbol{f}|_{-1,2}\right) \\ &\le C\left(\|\boldsymbol{u}\|_{\mathbf{4}}^{\mathbf{2}} + |\boldsymbol{f}|_{-1,2}\right) \end{aligned}$$

Recall $X(\Omega) \subset L^4(\Omega)$ and

$$\|\boldsymbol{u}\|_4 \leq C\, |\partial_1 \boldsymbol{u}|_{-1,2}^{\frac{1}{4}} \|\nabla \boldsymbol{u}\|_2^{\frac{3}{4}}$$

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Thus

$$\begin{aligned}
\|\nabla \boldsymbol{u}\|_2 + |\partial_1 \boldsymbol{u}|_{-1,2} &\leq C\left(\|\boldsymbol{u}\|_{\mathbf{4}}^{\mathbf{2}} + |\boldsymbol{f}|_{-1,2}\right) \\
&\leq C\left(|\partial_{\mathbf{1}} \boldsymbol{u}|_{-\mathbf{1},\mathbf{2}}^{\frac{\mathbf{1}}{\mathbf{2}}}\|\nabla \boldsymbol{u}\|_{\mathbf{2}}^{\frac{\mathbf{3}}{\mathbf{2}}} + |\boldsymbol{f}|_{-1,2}\right) \\
&\leq \tfrac{1}{2}|\partial_1 \boldsymbol{u}|_{-1,2} + C\left(\|\nabla \boldsymbol{u}\|_2^3 + |\boldsymbol{f}|_{-1,2}\right) \\
\Longrightarrow \|\nabla \boldsymbol{u}\|_2 + |\partial_1 \boldsymbol{u}|_{-1,2} &\leq C\left(|\boldsymbol{f}\|_{-1,2}^3 + |\boldsymbol{f}|_{-1,2}\right)
\end{aligned}$$

QED

THEOREM (Existence, GPG '07).

Steady-State Solutions: Generic Properties and Bifurcation

THEOREM (Existence, GPG '07). Let $\lambda > 0$. For any $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$ there is at least one

$$\boldsymbol{u} \in X(\Omega)$$

such that

$$\left.\begin{aligned} &\Delta\boldsymbol{u} - \lambda(\partial_1\boldsymbol{u} + \boldsymbol{V}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{V}) - \nabla p \\ &\qquad = \lambda\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \boldsymbol{H} + \boldsymbol{f} \\ &\operatorname{div}\boldsymbol{u} = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\boldsymbol{u}(x) = \boldsymbol{0}\,, \quad \boldsymbol{x} \in \partial\Omega\,, \quad \lim_{|x|\to\infty} \boldsymbol{u}(x) = \boldsymbol{0}$$

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Moreover, for "almost all" $\boldsymbol{f}$, the corresponding number of solutions is odd.

THEOREM (Global Uniqueness, GPG '94).

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There is $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$ the steady-state problem (SSP)

$$\left.\begin{aligned} &\Delta \boldsymbol{u} - \lambda(\partial_1 \boldsymbol{u} + \boldsymbol{V} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{V}) - \nabla p \\ &\qquad = \lambda \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \boldsymbol{H} \\ &\operatorname{div} \boldsymbol{u} = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\boldsymbol{u}(x) = \boldsymbol{0}\,, \quad \boldsymbol{x} \in \partial\Omega\,, \quad \lim_{|x|\to\infty} \boldsymbol{u}(x) = \boldsymbol{0}$$

has only one solution $\boldsymbol{u} \in X(\Omega)$.

Steady-State Solutions: Generic Properties and Bifurcation

$\lambda_0 = \sup\{\lambda : \text{SSP has unique solution } \boldsymbol{u}(\lambda) \in X(\Omega)\}$

What happens for $\lambda > \lambda_0$?

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What happens for $\lambda > \lambda_0$?
Let $\boldsymbol{u}_0 \in X(\Omega)$ the solution to SSP corresponding to $\lambda = \lambda_0$.

Recall that the linearization at $(\lambda_0, \boldsymbol{u}_0)$:

$$\mathcal{M}_{\boldsymbol{u}_0,\lambda_0} \colon \boldsymbol{w} \in X(\Omega) \mapsto \underbrace{\Delta \boldsymbol{w} - \lambda_0\, \partial_1 \boldsymbol{w} + \lambda_0 (\boldsymbol{V} \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{V})}_{\text{homeomorphism}}$$

$$-\lambda_0 \underbrace{(\boldsymbol{u}_0 \cdot \nabla \boldsymbol{w} + \nabla \boldsymbol{w} \cdot \nabla \boldsymbol{u}_0)}_{\text{compact operator}} \in \mathcal{D}_0^{-1,2}(\Omega)$$

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$$\mathcal{M}_{\boldsymbol{u}_0,\lambda_0}(\boldsymbol{w}) = 0,\ \boldsymbol{w} \in X(\Omega)\,, \implies \boldsymbol{w} = \mathbf{0}.$$

Then, there is $U(\lambda_0)$ such that for all $\lambda \in U(\lambda_0)$, SSP has a unique and analytic branch of solutions $\boldsymbol{u}(\lambda) \in X(\Omega)$ with $\boldsymbol{u}(\lambda_0) = \boldsymbol{u}_0$.

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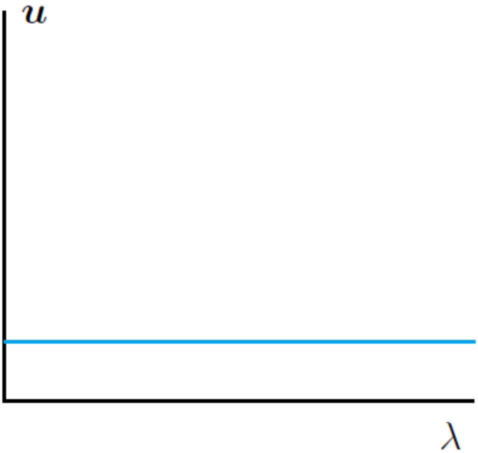
Remark. The branch can be continued up to the first value λ_1 where there is $\boldsymbol{w}^* \in X(\Omega) - \{\boldsymbol{0}\}$:

$$\mathcal{M}_{\boldsymbol{u}_1,\lambda_1}(\boldsymbol{w}^*) = 0$$

Steady-State Solutions: Generic Properties and Bifurcation

Possible Scenario



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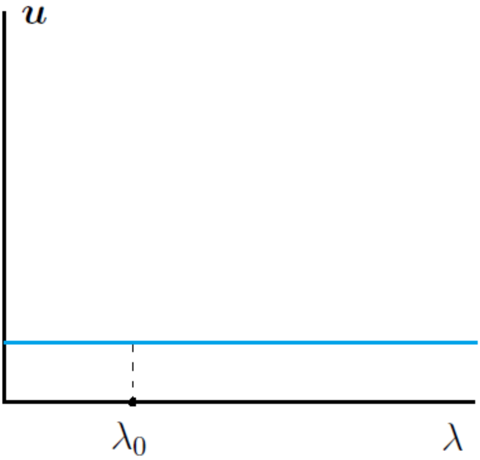

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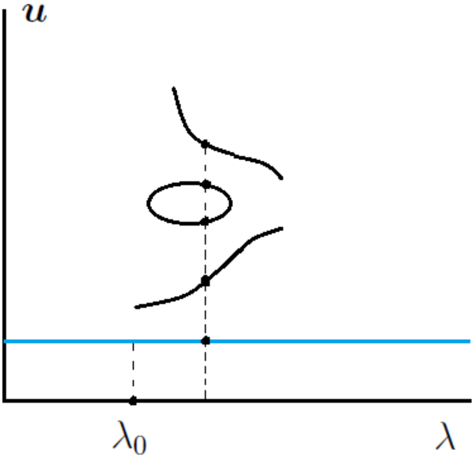

Steady-State Solutions: Generic Properties and Bifurcation

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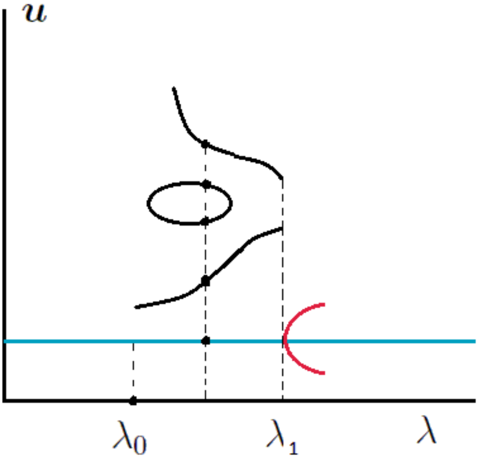

Steady-State Solutions: Generic Properties and Bifurcation

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$$
\begin{aligned}
M : (\lambda, \boldsymbol{x} := \ & (x_1, x_2)) \in \mathbb{R} \times \mathbb{R}^2 \\
& \mapsto \begin{pmatrix} x_1(1-\lambda) - x_2 \left|\boldsymbol{x}\right|^2 \\ x_2(1-\lambda) + x_1 \left|\boldsymbol{x}\right|^2 \end{pmatrix} \in \mathbb{R}^2 \, .
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\end{aligned}
$$

The linearization at $\boldsymbol{x} = 0$ *has a nontrivial solution* (only) at $\lambda = 1$. However, the equations

$$x_1(1-\lambda) - x_2 \, |\boldsymbol{x}|^2 = 0 \, , \;\; x_2(1-\lambda) + x_1 \, |\boldsymbol{x}|^2 = 0$$

have only the solution $x_1 = x_2 = 0$ for *any* $\lambda \in \mathbb{R}$.

Sufficient condition for bifurcation from $(\lambda_1, \boldsymbol{u}_1)$.

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Let $\overline{\boldsymbol{u}}(\lambda)$, $\lambda \in U(\lambda_1)$, be a sufficiently smooth branch with $\overline{\boldsymbol{u}}(\lambda_1) = \boldsymbol{u}_1$. *For simplicity,* $\overline{\boldsymbol{u}} \equiv \boldsymbol{u}_1$, *for all* $\lambda \in U(\lambda_1)$.

Sufficient condition for bifurcation from $(\lambda_1, \boldsymbol{u}_1)$.

Let $\overline{\boldsymbol{u}}(\lambda)$, $\lambda \in U(\lambda_1)$, be a sufficiently smooth branch with $\overline{\boldsymbol{u}}(\lambda_1) = \boldsymbol{u}_1$. *For simplicity,* $\overline{\boldsymbol{u}} \equiv \boldsymbol{u}_1$*, for all* $\lambda \in U(\lambda_1)$. Setting $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}_1$, we find

$$\begin{aligned}&\Delta \boldsymbol{w} - \lambda(\partial_1 \boldsymbol{w} + \boldsymbol{u}_1 \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{u}_1 + \boldsymbol{w} \cdot \nabla \boldsymbol{w}) = \nabla p\\ &\operatorname{div} \boldsymbol{w} = 0\end{aligned}$$

$$\boldsymbol{w} = \mathbf{0} \text{ at } \partial\Omega\,.$$

It is enough to show the existence of $\boldsymbol{w}(\lambda) \in X(\Omega)$, $\lambda \in U(\lambda_1)$:

$$\boldsymbol{w}(\lambda) \not\equiv \mathbf{0}\,, \quad \boldsymbol{w}(\lambda) \to \mathbf{0} \text{ as } \lambda \to \lambda_1\,.$$

The crucial property that allows us to provide sufficient condition for the occurrence of (steady) bifurcation is that the operator

$$L : \boldsymbol{w} \in X(\Omega) \mapsto \Delta \boldsymbol{w} - \lambda_1 (\partial_1 \boldsymbol{w} + \boldsymbol{u}_1 \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{u}_1) \in \mathcal{D}_0^{-1,2}(\Omega)$$

is Fredholm of index 0.

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(A) Nontrivial linearization The problem

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(B) Branching Condition The problem

$$\Delta \boldsymbol{w} - \lambda_1(\partial_1 \boldsymbol{w} + \boldsymbol{u}_1 \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{u}_1) = \nabla p - \frac{1}{\lambda_1} \Delta \boldsymbol{w}_1$$
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has no solution $\boldsymbol{w} \in X(\Omega)$.

An Important Remark. Condition (A) does *not* mean that 0 is an eigenvalue (of geometric multiplicity 1) of the operator

$$L : \boldsymbol{w} \in X(\Omega) \mapsto \Delta \boldsymbol{w} - \lambda_1(\partial_1 \boldsymbol{w} + \boldsymbol{u}_1 \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{u}_1) \in \mathcal{D}_0^{-1,2}(\Omega) \; .$$

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When *instead* defined in $W^{2,2}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega)$ with values in $L^2(\Omega)$ the linearized differential operator $\widetilde{L}$ (say) has a well-studied spectrum (Babenko '82, Neustupa '06).

In particular, the spectrum of $\widetilde{L}$ may contain isolated eigenvalues of finite multiplicity.

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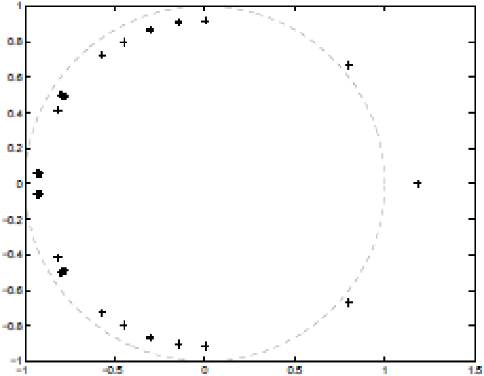

Distributions of the Eigenvalues in Flow past a Sphere

J. Dušek et al. (2011)

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On the other hand, the operator L is Fredholm of index 0.

However, we can still rephrase our bifurcation theorem in terms of the spectrum of a suitable linearized operator.

THEOREM 1 (GPG '07) $(\lambda_1, \boldsymbol{u}_1)$ is a bifurcation point if:

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A sufficient condition for the validity of (A) & (B) is given in terms of the *spectrum*, $\mathrm{Sp}(\mathcal{L})$, of the linearized operator:

$$\mathcal{L} : \boldsymbol{w} \in X(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega) \mapsto \Delta^{-1}(\partial_1 \boldsymbol{w} + \boldsymbol{u}_1 \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{u}_1) \in \mathcal{D}_0^{1,2}(\Omega)\,.$$

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Lemma (GPG '07) $\mathcal{L}$ is closed and $\mathrm{Sp}(\mathcal{L}) \cap (0, \infty)$ consists of an at most countable number of eigenvalues of finite algebraic multiplicity clustering only at 0.

THEOREM 2 (GPG '07) Sufficient condition for $(\lambda_1, \boldsymbol{u}_1)$ to be a bifurcation point is that $1/\lambda_1$ is eigenvalue with algebraic multiplicity 1 (simple eigenvalue) of the operator $\mathcal{L}$.

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How many bifurcation points for Reynolds number in a finite interval?

THEOREM 3 (GPG '07)

Let $\boldsymbol{u}_1 \in X(\Omega)$ be a solution branch for $\lambda \in J$, where J is a bounded interval with $\overline{J} \in (0, \infty)$. Then, there is at most a finite numbers of bifurcation points $(\lambda_k, \boldsymbol{u}_1)$, $\lambda_k \in J$, $k = 1, 2, \ldots, m$.

Possible Scenario for the Solution Manifold

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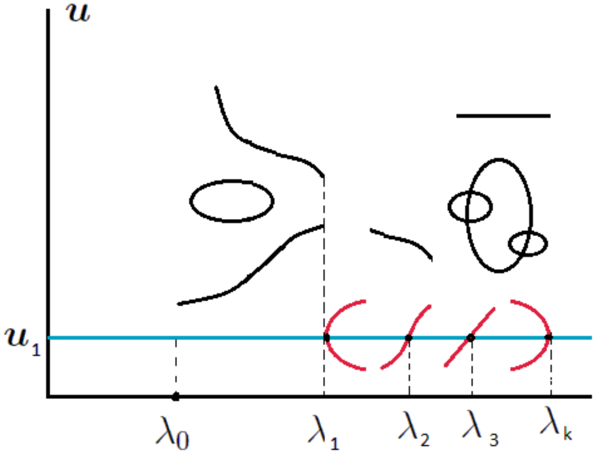
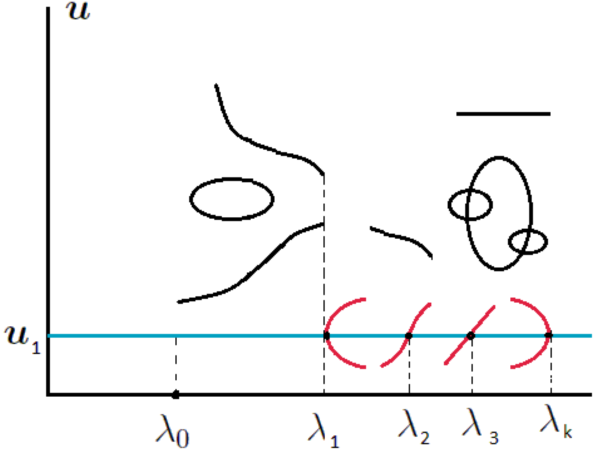


Steady-State Solutions: Generic Properties and Bifurcation

Possible Scenario for the Solution Manifold



Generically, the manifold cannot look like this!

THEOREM 4 (GPG '10)

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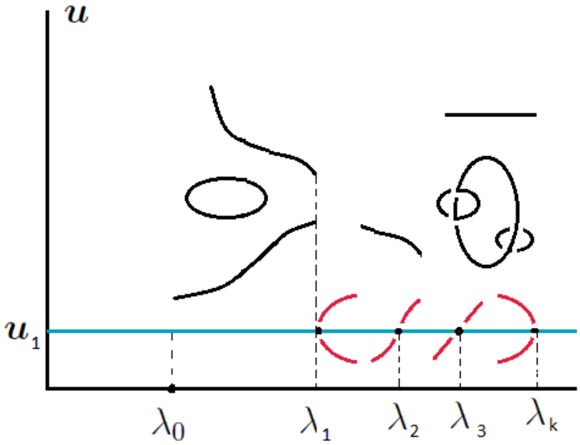
For "almost all" $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ the solution manifold

$$\mathfrak{M}(\boldsymbol{f}) = \{(\lambda, \boldsymbol{u}) \in X(\Omega) : \mathcal{M}(\lambda, \boldsymbol{u}) = \boldsymbol{f}\}$$

is a C^∞ 1-dimensional (Banach) manifold.

Generic Scenario for the Solution Manifold



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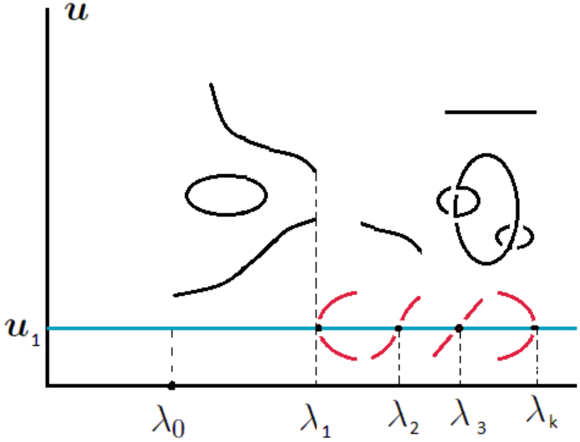

Generically, steady bifurcation cannot occur!

Steady-State Solutions: Time-Periodic Bifurcation

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Let $\boldsymbol{u}_0 \in X(\Omega)$ be a steady-state solution at $\lambda = \lambda_0$ and suppose $(\lambda_0, \boldsymbol{u}_0)$ is *not* a steady-state bifurcation point (linearization is trivial):

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$$\left.\begin{aligned} \Delta \boldsymbol{w} + \lambda_0 \left(\partial_1 \boldsymbol{w} - \boldsymbol{u}_0 \cdot \nabla \boldsymbol{w} - \boldsymbol{w} \cdot \nabla \boldsymbol{u}_0\right) \\ +\nabla p = 0 \\ \boldsymbol{w} \in X(\Omega) \end{aligned}\right\} \Longrightarrow \boldsymbol{w} \equiv \boldsymbol{0}$$

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Then, there is a unique analytic family of steady-state solutions $\boldsymbol{u}(\lambda) \in X(\Omega)$, $\lambda \in U(\lambda_0)$, with $\boldsymbol{u}(\lambda_0) = \boldsymbol{u}_0$. For simplicity, we assume

$$\boldsymbol{u}(\lambda) = \boldsymbol{u}_0\,, \quad \text{all } \lambda \in U(\lambda_0).$$

Steady-State Solutions: Time-Periodic Bifurcation

Roughly, the time-periodic bifurcation problem consists in finding a time-periodic solution in any neighborhood of the point $(\lambda_0, \boldsymbol{u}_0)$.

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Writing $\boldsymbol{U} = \boldsymbol{u}_0 + \boldsymbol{V} + \boldsymbol{v}$ in the original equations

$$\left.\begin{aligned} \partial_t \boldsymbol{U} + \lambda \boldsymbol{U} \cdot \nabla \boldsymbol{U} = \Delta \boldsymbol{U} - \nabla p \\ \operatorname{div} \boldsymbol{U} = 0 \end{aligned}\right\} \text{ in } \Omega \times (0, \infty)$$

$$\boldsymbol{U}(x,t) = \mathbf{0}\,, \quad \boldsymbol{x} \in \partial\Omega\,, \quad \lim_{|x|\to\infty} \boldsymbol{U}(x,t) = \boldsymbol{e}_1\,, \quad t \geq 0$$

the problem can be more precisely formulated as follows

Find a family of non-trivial time-periodic functions $\boldsymbol{v}(\lambda)$, of period $T = T(\lambda)$, $\lambda \in U(\lambda_0)$, such that

$$\left.\begin{aligned} \partial_t \boldsymbol{v} - \lambda\big(\partial_1 \boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u}_0 - \boldsymbol{u}_0\cdot\nabla\boldsymbol{v}\big) & \\ + \boldsymbol{v}\cdot\nabla\boldsymbol{v} &= \Delta\boldsymbol{v} - \nabla\phi \\ \operatorname{div}\boldsymbol{v} &= 0 \end{aligned}\right\} \text{in } \Omega\times\mathbb{R}$$

$$\boldsymbol{v} = \boldsymbol{0} \ \text{ at } \partial\Omega\times\mathbb{R}\,, \quad \lim_{|x|\to\infty}\boldsymbol{v}(x,\tau) = \boldsymbol{0}\,, \quad \tau\in\mathbb{R}\,,$$

with $\boldsymbol{v}(\lambda) \to \boldsymbol{0}$ as $\lambda \to \lambda_0$.

Steady-State Solutions: Time-Periodic Bifurcation

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with $\boldsymbol{v}(\lambda) \to \mathbf{0}$ as $\lambda \to \lambda_0$.

(1) $\boldsymbol{v} \equiv \mathbf{0}$ is a solution for all $\lambda \in U(\lambda_0)$;

(2) The frequency, $\omega(\lambda) := 2\pi/T(\lambda)$, is unknown .

Steady-State Solutions: Time-Periodic Bifurcation

Self-Oscillation in Flow Past a Cylinder

Steady-State Solutions: Time-Periodic Bifurcation

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The problem then reduces to find a family of non-trivial 2π-periodic solutions $\boldsymbol{v}(\lambda)$, $\lambda \in U(\lambda_0)$, such that

$$\left.\begin{aligned} \omega\,\partial_\tau \boldsymbol{v} - \lambda\Big[\big(\partial_1 \boldsymbol{v} - \boldsymbol{v}\cdot\nabla \boldsymbol{u}_0 - \boldsymbol{u}_0\cdot\nabla \boldsymbol{v}\big) & \\ +\boldsymbol{v}\cdot\nabla\boldsymbol{v}\Big] &= \Delta \boldsymbol{v} - \nabla\phi \\ \operatorname{div}\boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \Omega\times\mathbb{R}$$

$$\boldsymbol{v} = \boldsymbol{0} \ \text{ at } \partial\Omega\times\mathbb{R}\,, \quad \lim_{|x|\to\infty} \boldsymbol{v}(x,t) = \boldsymbol{0}\,, \quad t\in\mathbb{R}\,,$$

with $\boldsymbol{v}(\lambda) \to \boldsymbol{0}$, as $\lambda \to \lambda_0$.

The next step is to have an "approximated" 2π-periodic solution to start with.

Steady-State Solutions: Time-Periodic Bifurcation

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Consider the linear operator at "criticality" $(\lambda = \lambda_0)$

$$\mathcal{L}_0(\boldsymbol{v}) := \mathrm{P}\Big[\Delta \boldsymbol{v} + \lambda_0\big(\partial_1 \boldsymbol{v} - \boldsymbol{u}_0 \cdot \nabla \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \boldsymbol{u}_0\big)\Big],$$

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Let $\boldsymbol{a}$ be the unique (normalized) eigenvector associated to the eigenvalue $\mathrm{i}\,\omega_0$.

Then,

$$\boldsymbol{v}_1 := \Re[\boldsymbol{a}\, e^{i\,\tau}]\,, \quad \boldsymbol{v}_2 := \Im[\boldsymbol{a}\, e^{i\,\tau}]$$

are 2π-periodic solutions to the linearized problem

$$\left.\begin{aligned} &\omega_0\, \partial_\tau \boldsymbol{v} - \mathrm{P}\big[\lambda_0(\partial_1 \boldsymbol{v} - \boldsymbol{v}\cdot\nabla \boldsymbol{u}_0 - \boldsymbol{u}_0\cdot\nabla\boldsymbol{v}) + \Delta\boldsymbol{v}\big] = 0 \\ &\operatorname{div} \boldsymbol{v} = 0 \end{aligned}\right\}$$

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The basic idea is then to construct a solution to the full nonlinear problem "around" the time-periodic solution $\boldsymbol{v}_1$ (say) of the above linearization.

Steady-State Solutions: Time-Periodic Bifurcation

To use a perturbative argument, we write:

$$\left.\begin{aligned}
&\omega_0\,\partial_\tau \boldsymbol{v} - \lambda_0(\partial_1 \boldsymbol{v} - \boldsymbol{v}\cdot\nabla \boldsymbol{u}_0 - \boldsymbol{u}_0\cdot\nabla\boldsymbol{v}) + \Delta\boldsymbol{v}\\
&\qquad\qquad\qquad\qquad = -\nabla\phi + \boldsymbol{N}(\mu,\delta,\boldsymbol{v})\\
&\operatorname{div}\boldsymbol{v} = 0
\end{aligned}\right\}$$

$$\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\,, \quad \lim_{|x|\to\infty}\boldsymbol{v}(x,t) = \boldsymbol{0}$$

with

$$\boldsymbol{N} := -\mu\,\partial_\tau\boldsymbol{v} + \delta\,[\partial_1\boldsymbol{v} - \boldsymbol{u}_0\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u}_0] - \lambda\boldsymbol{v}\cdot\nabla\boldsymbol{v}\,;$$

$$\mu := \omega - \omega_0\,;\quad \delta := \lambda - \lambda_0\,,$$

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$$\mu := \omega - \omega_0\,; \quad \delta := \lambda - \lambda_0\,,$$

and require

$\operatorname{proj}(\boldsymbol{v}) = \varepsilon$, all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$; $\operatorname{proj} : \boldsymbol{v} \mapsto \operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$

Steady-State Solutions: Time-Periodic Bifurcation

We frame the problem in B-spaces $\mathcal{Y}, \mathcal{X}(\subseteq \mathcal{Y})$:

$$\omega_0 \frac{dv}{d\tau} - L(v) = N(\mu, \delta, v)\,, \quad \text{in } \mathcal{Y}\,; \;\; \mathrm{proj}\,(v) = \varepsilon\,,$$

where

$$L : \mathcal{X} \to \mathcal{Y}\,; \quad N : (\mu, \delta, v) \in \mathbb{R}^2 \times \mathcal{X} \to \mathcal{Y}$$

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Existence of a 2π-periodic branch in ε follows from the IFT, if

$$\omega_0 \frac{d}{d\tau} - L$$

has a bounded inverse in a (suitable) class of 2π-periodic functions.

Steady-State Solutions: Time-Periodic Bifurcation

How do we choose the spaces appropriately?

How do we choose the spaces appropriately?

The "classical" approach (Iudovich, Sattinger, Joseph, Iooss...) requires

$$\mathcal{X} \equiv W^{2,2}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega)\,,$$

$$\mathcal{Y} \equiv H(\Omega) := \left\{ \boldsymbol{u} \in L^2(\Omega) : \ \operatorname{div} \boldsymbol{u} = 0\,, \ \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \right\}.$$

This choice is suitable for flow in bounded region, but it is not right in the case at hand.

Steady-State Solutions: Time-Periodic Bifurcation

Actually, bounded invertibility of

$$\omega_0 \frac{d}{d\tau} - L$$

in a class of 2π-periodic functions requires, in particular, bounded invertibility of L.

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For flow in bounded domain, the operator L, defined on $\mathcal{X} \equiv W^{2,2}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega)$ has a purely discrete spectrum $\mathrm{Sp}_p(L)$, so it is enough to assume $0 \notin \mathrm{Sp}_p(L)$.

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For flow past an obstacle, 0 is in the essential spectrum of L (Babenko '82, Neustupa '06) and bounded invertibility is no longer guaranteed.

Steady-State Solutions: Time-Periodic Bifurcation

$$v = \overline{v} + (v - \overline{v}) := \overline{v} + w\,; \quad \overline{v} := \tfrac{1}{2}\int_{-\pi}^{\pi} v(t)\,\mathrm{d}t$$

($\overline{v} =$ average and $w =$ oscillatory component). Then,

$$\omega_0 \frac{dv}{d\tau} - L(v) = N(\mu, \delta, v)\,, \ \text{in } \mathcal{Y}\,, \ v(\tau) = v(\tau + 2\pi)\,,$$

is split as a coupled "elliptic-parabolic" system

$$L_1(\overline{v}) = N_1(\mu, \delta, \overline{v}, w)$$

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$$L_1 : \boldsymbol{w} \in X(\Omega) \mapsto$$

$$\Delta\boldsymbol{w} - \lambda_0(\partial_1\boldsymbol{w} + \boldsymbol{u}_0 \cdot \nabla\boldsymbol{w} + \boldsymbol{w} \cdot \nabla\boldsymbol{u}_0) \in \mathcal{D}_0^{-1,2}(\Omega)$$

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$$L_2 : \boldsymbol{w} \in W^{2,2}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega) \mapsto$$

$$\Delta \boldsymbol{w} - \lambda_0(\partial_1 \boldsymbol{w} + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{u}_0) \in H(\Omega)$$

Steady-State Solutions: Time-Periodic Bifurcation

Since L_1 is Fredholm of index 0, bounded invertibility is equivalent to

$$L_1(\boldsymbol{w}) = \mathbf{0} \Longrightarrow \boldsymbol{w} = \mathbf{0}$$

that is, $(\lambda_0, \boldsymbol{u}_0)$ is not a steady bifurcation point.

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As for L_2, we have the following:

Lemma $\mathrm{Sp}(L_2) \cap \{\mathrm{i}\,\mathbb{R} - \{0\}\}$ is bounded and constituted by a countable number of isolated eigenvalues of finite algebraic multiplicity (a.m.) that can only cluster at 0.

ASSUMPTIONS

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(H1) $L_1(\boldsymbol{w}) = \mathbf{0}\,,\ \boldsymbol{w} \in X(\Omega)\,, \quad \Longrightarrow \quad \boldsymbol{w} = \mathbf{0}$

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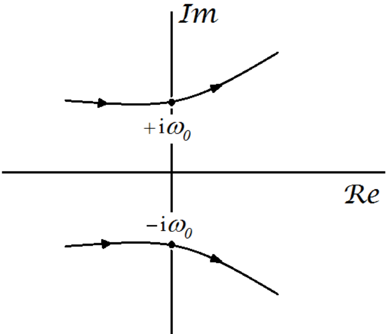
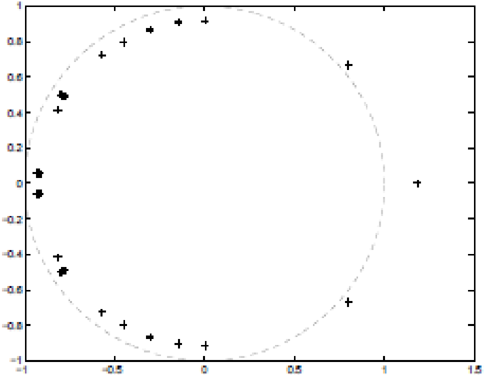
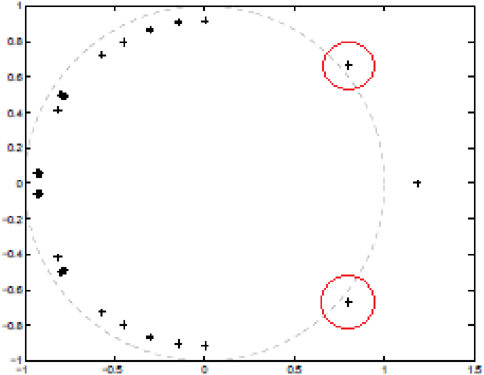


Steady-State Solutions: Time-Periodic Bifurcation



Distributions of the Eigenvalues
J. Dušek et al. (2011)

Steady-State Solutions: Time-Periodic Bifurcation



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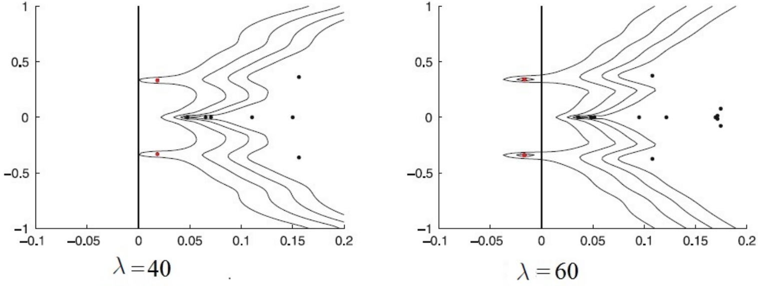

Distributions of the Eigenvalues in 2D
R. Rannacher et al. (2012)

THEOREM 5 (GPG '16)

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$$\begin{aligned}\boldsymbol{U}(x,\tau;\lambda(\varepsilon)) = \; & \boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{V}(x;\lambda_0) \\ & +\varepsilon\left[(\cos\tau)\boldsymbol{A}_1 + (\sin\tau)\boldsymbol{A}_2\right] + O(\varepsilon^2)\,,\end{aligned}$$

with $\boldsymbol{A}_i \in W^{2,2}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega)$.

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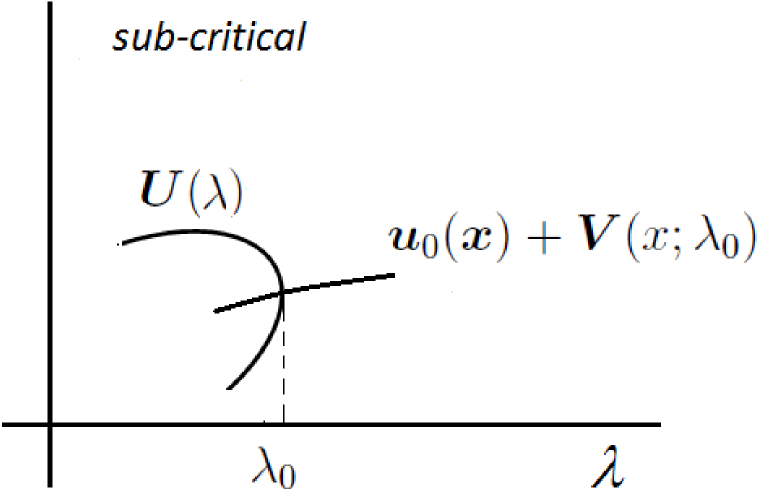
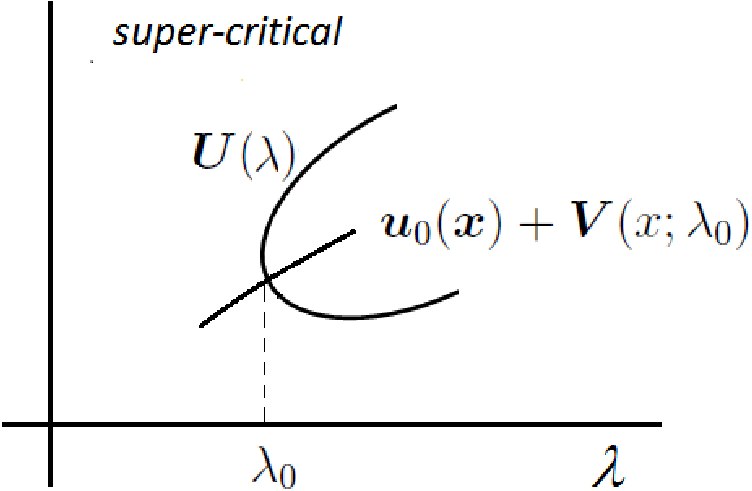
with $\boldsymbol{A}_i \in W^{2,2}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega)$.

(C) Bifurcation is either sub- or super-critical.

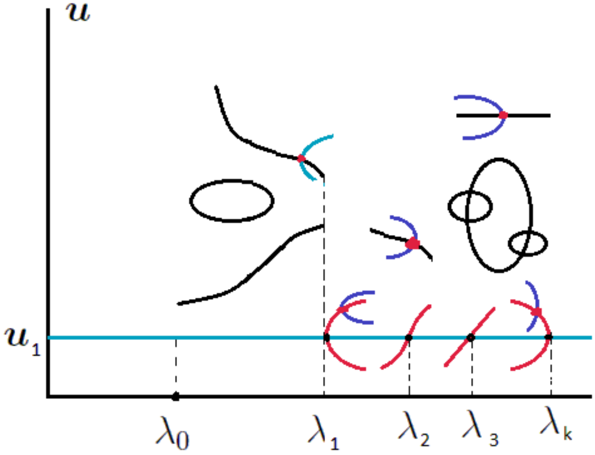
Steady-State Solutions: Time-Periodic Bifurcation



Steady-State Solutions: Time-Periodic Bifurcation



Updated Scenario for the Solution Manifold



An Important Question to Investigate

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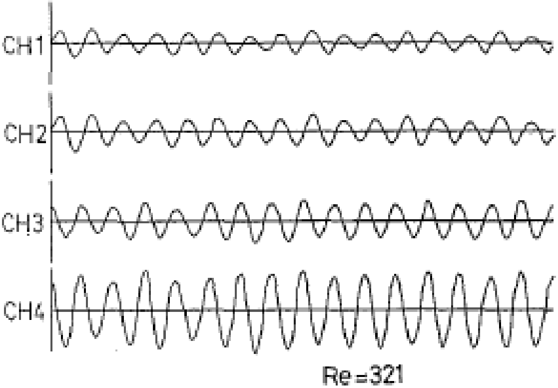
It is experimentally observed that in the region $320 \lesssim \lambda \lesssim 500$ there is a flow transition where the oscillations are no longer monochromatic but may involve, instead, a finite number of modes with a *larger* period.

Steady-State Solutions: Time-Periodic Bifurcation

Experimental Finding



Waveform of Fluctuating Velocity in the Wake

Sakamoto & Haniu (1990)

Steady-State Solutions: Time-Periodic Bifurcation

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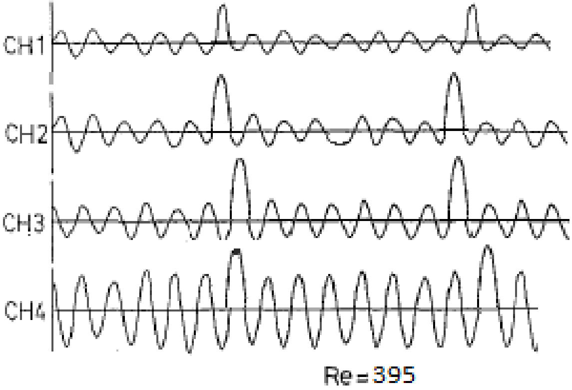

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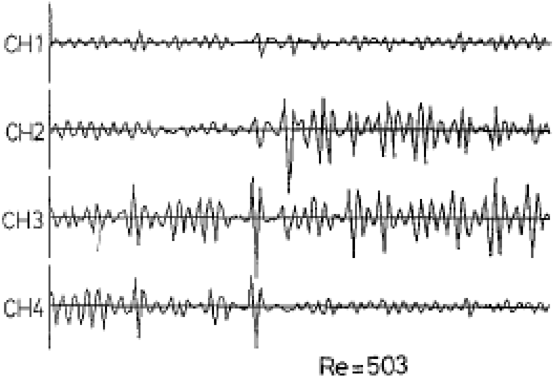

Waveform of Fluctuating Velocity in the Wake

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Steady-State Solutions: Time-Periodic Bifurcation

From the mathematical viewpoint, this means to study time-periodic bifurcation from a time-periodic flow.

While this problem has been widely studied in the case of flow in a *bounded* domain (Ruelle & Takens, Marsden & McCracken, Iooss, Iooss & Joseph, ...) it appears to be very complicated for a flow past an obstacle.

Stability and Long-Time Behavior

The Stability Problem.

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Which one among the great variety of solutions is stable (and hence physically observable)? This is a formidable question.

A sufficiently complete answer is only available for the steady-state *laminar* solution, that is, the one that exists for all $\lambda > 0$. No rigorous result is, instead, available for stability of *bifurcating* solutions.

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Let

$$u_0 = u_0(\lambda) \in X(\Omega)\,, \;\; \lambda > 0$$

be the laminar steady-state solution.

Stability and Long-Time Behavior

A field $\boldsymbol{v} \in L^\infty(0,T;H(\Omega)) \cap L^2(0,T;\mathcal{D}_0^{1,2}(\Omega))$ all $T>0$ is in the Leray-Hopf class if satisfies

1. The “perturbation equation” (in the distributions sense):

$$\frac{\partial \boldsymbol{v}}{\partial t} + \lambda(\boldsymbol{e}_1 + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{u}_0) = -\lambda \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \Delta \boldsymbol{v} - \nabla p$$

2. The Strong Energy Inequality:

$$\|\boldsymbol{v}(t)\|_2^2 \leq \|\boldsymbol{v}(s)\|_2^2 - 2\int_s^t \left[\lambda(\boldsymbol{v}\cdot\nabla \boldsymbol{u}_0, \boldsymbol{v}) + \|\nabla \boldsymbol{v}\|_{1,2}^2\right] d\tau\,,$$

a.a. $s>0$ (including $s=0$) and all $t \in [s,T]$.

THEOREM (Maremonti '85)

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$$\lambda_* := \sup_{\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)} \frac{-(\boldsymbol{\varphi} \cdot \nabla \boldsymbol{u}_0 \cdot \boldsymbol{\varphi})}{\|\nabla \boldsymbol{\varphi}\|_2^2} .$$

Then, if

$$\lambda < \lambda_*$$

all "perturbations" $\boldsymbol{v}$ in the Leray-Hopf class with $\boldsymbol{v}(0) \in H(\Omega)$ satisfy

$$\|\boldsymbol{v}(t)\|_2 \leq \|\boldsymbol{v}(0)\|_2 , \quad \lim_{t \to \infty} \|\boldsymbol{v}(t)\|_2 = 0$$

namely, $\boldsymbol{u}_0$ is asymptotically stable in the L^2–norm.

Stability and Long-Time Behavior

Maremonti's result (and many other related ones) requires "smallness" of the steady-state.

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Spectral Stability

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Spectral Stability

Assume all eigenvalues of the linearized operator

$$L_2 : \boldsymbol{w} \in W^{2,2}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega) \mapsto \Delta \boldsymbol{w} - \lambda_0 (\partial_1 \boldsymbol{w} + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{u}_0) \in H(\Omega)$$

have negative real part. Is the steady-state $\boldsymbol{u}_0$ asymptotically stable?

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have negative real part. Is the steady-state $\boldsymbol{u}_0$ asymptotically stable?

The answer is *positive* for flow in a *bounded* domain. What for flow past an obstacle?

Stability and Long-Time Behavior

THEOREM (Neustupa '99, '09, '10, '16)

Stability and Long-Time Behavior

THEOREM (Neustupa '99, '09, '10, '16) Let μ denote the generic eigenvalue of L_2, and suppose there exist $\delta_1, \delta_2 > 0$ such that

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(plus another "technical" condition).

There are $\eta, C > 0$ such that if

$$\|\boldsymbol{v}(0)\|_2 + \|\nabla \boldsymbol{v}(0)\|_2 < \eta,$$

then

$$\|\boldsymbol{v}(t)\|_2 + \|\nabla \boldsymbol{v}(t)\|_2 < C\eta\,, \ \text{all } t > 0\,;$$
$$\lim_{t\to\infty} \|\nabla \boldsymbol{v}(t)\|_2 = 0\,.$$

Stability and Long-Time Behavior

Fundamental Open Question:

Long-time Behavior for “Large” λ.

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Long-time Behavior for "Large" λ.

Given $\lambda > 0$ and $\boldsymbol{v}_0$, study the behavior as $t \to \infty$ of solutions $\boldsymbol{v}$ (in a suitable class) to:

$$\left.\begin{array}{l} \partial_t \boldsymbol{v} - \lambda(\partial_1 \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) = \Delta \boldsymbol{v} - \nabla \mathsf{p} \\ \operatorname{div} \boldsymbol{v} = 0 \end{array}\right\} \text{in } \Omega \times \mathbb{R}_+$$

$$\boldsymbol{v}(x,t)|_{\partial\Omega} = \boldsymbol{e}_1, \ \lim_{|x|\to\infty} \boldsymbol{v}(x,t) = \boldsymbol{0}\,, \ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x)\,.$$

Fundamental Open Question:
Long-time Behavior for "Large" λ.

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The basic difficulty is that the data ($\boldsymbol{e}_1$) are *time-independent*.

Stability and Long-Time Behavior

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The first question is: Is there a norm, $\| \cdot \|_{\mathfrak{X}}$, with respect to which solutions are uniformly bounded:

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Certainly,

$$\| \cdot \|_{\mathfrak{X}} \neq \| \cdot \|_2 \,,$$

that is, the kinetic energy is expected to be unbounded even for small λ .

Actually, assume $\lambda < \lambda_0$ and

$$\|\boldsymbol{v}(t)\|_2 \leq K\,, \quad K \text{ independent of } t\,. \tag{1}$$

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Then, there are unbounded sequence, $\{t_m\}$, and $\boldsymbol{v}^0 \in L^2(\Omega)$ (maybe depending on the sequence):

$$\lim_{m\to\infty}(\boldsymbol{v}(t_m),\boldsymbol{\varphi}) = (\boldsymbol{v}^0,\boldsymbol{\varphi})\,, \quad \text{for all } \boldsymbol{\varphi} \in C_0^\infty(\Omega)\,.$$

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Thus $\boldsymbol{v}^0 = \boldsymbol{v}_S$ (steady solution corresponding to λ). This gives $\boldsymbol{v}_S \in L^2(\Omega)$. However, it is well known that

$$\boldsymbol{v}_S \in L^q(\Omega)\,, \quad \text{for all } q > 2\,, \qquad \boldsymbol{v}_S \notin L^2(\Omega)$$

and (1) is not true.

One may thus try $\mathfrak{X} = L^q(\Omega)\,, \quad q \in (2,\infty)$.
However, the validity of the estimate

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Conjecture

$$\|\boldsymbol{v}(t)\|_q \le C(\lambda, \boldsymbol{v}_0)\,, \text{ for all } t > 0 \quad \text{some } q \in (2,3).$$

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Conjecture

$\|\boldsymbol{v}(t)\|_q \leq C(\lambda, \boldsymbol{v}_0)\,,$ for all $t > 0$ some $q \in (2, 3)$.

Remark. Proving the conjecture would be of *no harm* to the outstanding open problem of global regularity.

Stability and Long-Time Behavior

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Model Problem:

$$\left.\begin{aligned} &\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \Delta \boldsymbol{v} - \nabla p \\ &\operatorname{div} \boldsymbol{v} = 0 \end{aligned}\right\} \quad \text{in} \;\; \mathbb{R}^3 \times (0, \infty)$$

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Can we show

$\boldsymbol{v}_0 \in L^q(\Omega) \implies$ existence of $\boldsymbol{v} \in L^\infty(0,\infty; L^q(\mathbb{R}^3))$,

some $q \in (2,3)$.

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Let $\psi \in W^{k,\infty}(0,\infty), \quad k \geq 0$ with

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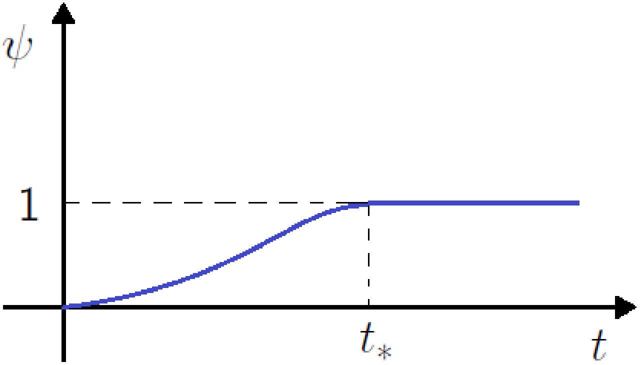
Does the problem possess a uniformly bounded (in time), global solution? A (local) attractor?

When ψ is a "ramp function" becoming 1 after $t = t_*$



Stability and Long-Time Behavior

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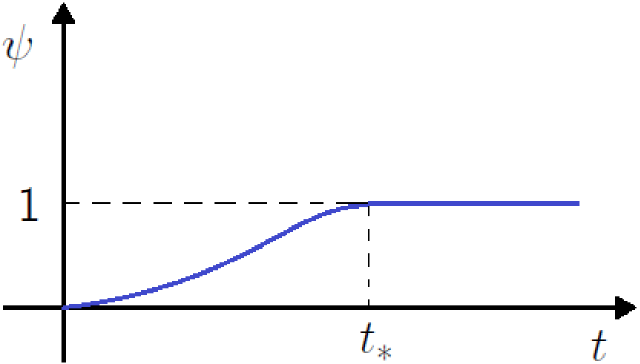
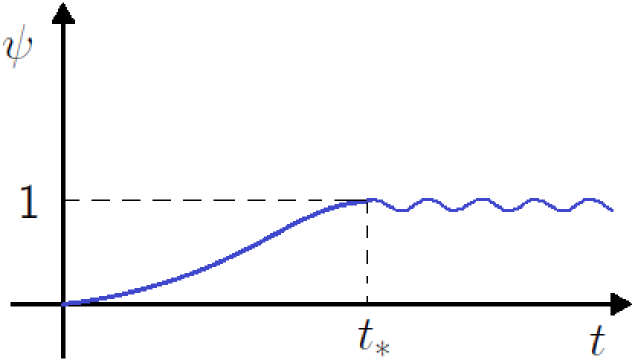


the solution exists, and tends to the uniquely determined steady-state flow (Heywood, Shibata, GPG '96).

When ψ is a "ramp function" becoming sinusoidal after $t = t_*$



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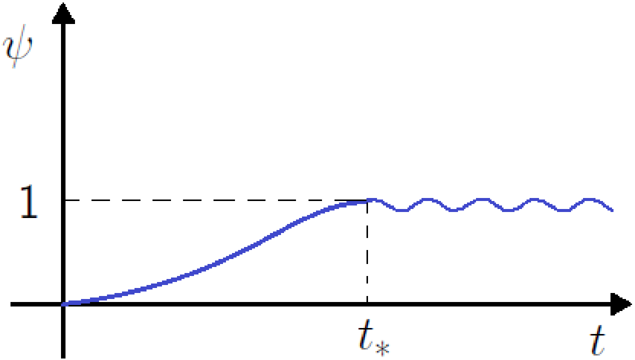
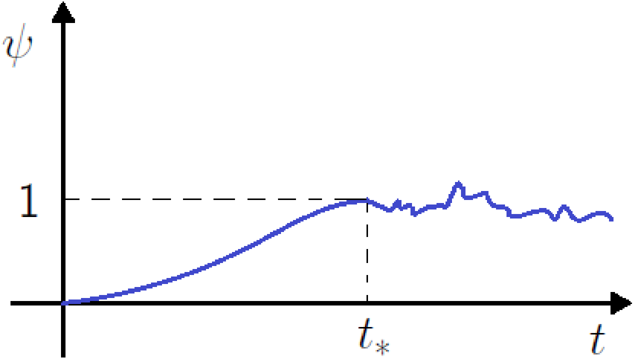


probably, the solution will tend to a time-periodic flow. However, this is open.

The "general" case:



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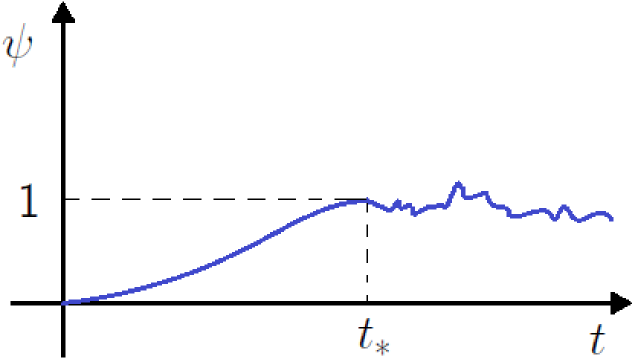

is, of course, entirely open.

Two Final Puzzles

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In the 2D case –planar flow past a cylinder– even the "simple" existence problem of a steady-state solution for *arbitrary* Reynolds number is an outstanding, long-lasting *open question* (Leray '33).

LIOUVILLE, LIOUVILLE are you there?

In the 2D case –planar flow past a cylinder– even the "simple" existence problem of a steady-state solution for *arbitrary* Reynolds number is an outstanding, long-lasting *open question* (Leray '33). That is, it is not known whether

$$\left.\begin{array}{c} \Delta \boldsymbol{v} - \lambda\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \nabla p \\ \operatorname{div} \boldsymbol{v} = 0 \end{array}\right\} \text{ in } \Omega$$

$$\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\,, \quad \lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{e}_1\,,$$

has a solution (in any "reasonable" function class) for *all* $\lambda > 0$.

Two Final Puzzles

The difficult part is to show that the constructed solution satisfies *also* the condition at infinity:

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{e}_1 \,.$$

The latter can be verified (to date) only for "small" λ (Finn & Smith '67, GPG '93)

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To add more interest to the problem, one *can* show that, in the case of "symmetric" flow, and arbitrary λ, there is $\alpha \in [0,1]$ such that (Amick '88, GPG '04)

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \alpha\boldsymbol{e}_1\,.$$

Two Final Puzzles

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Let $\Omega = \{x \in \mathbb{R}^2 : \ |x| > 1\}$, and let $\boldsymbol{u} \in C^\infty(\Omega)$ solve the homogeneous problem:

$$\left.\begin{aligned} \Delta \boldsymbol{u} &= \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla \phi \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}$$

$$\|\nabla \boldsymbol{u}\|_2 < \infty\,,$$

$$\lim_{|x|\to\infty} D^\alpha \boldsymbol{u}(x) = \boldsymbol{0} \ \text{ uniformly pointwise, all } |\alpha| \geq 0\,.$$

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Equivalently, if the homogeneous problem possesses a nonzero solution, then there is $\lambda_1 > 0$ such that the 2D exterior steady-state problem does not have a solution in any (reasonable) class for all $\lambda > \lambda_1$.

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Remark. Notice that if $\Omega \equiv \mathbb{R}^2$ the proof of the conjecture is well-known, trivial and useless:

$$\Delta\omega - \boldsymbol{u}\cdot\nabla\omega = \mathbf{0} \Longrightarrow \omega = \mathbf{0}\,.$$

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Another LIOUVILLE problem.

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Consider the equations

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Notice that, by Sobolev,

$$\nabla \boldsymbol{v} \in L^2(\mathbb{R}^3) \implies \boldsymbol{v} \in L^6(\mathbb{R}^3)\,.$$

Two Final Puzzles

I formulated this problem back in 1994.
Its resolution has gained more popularity after the paper of Seregin, Šverák *et al.* (2009), where it is shown that a finite-time singularity arising from a mild solution to the IVP generates a non-identically zero solution in $L^\infty((-\infty, 0) \times \mathbb{R}^3)$. (*Ancient Solution*)

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Since a steady-state solution is a particular ancient solution, giving a *negative* answer to the Liouville problem may provide valuable information to the notorious regularity question.

Two Final Puzzles

Some Available Main Results.

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CHAE '14: $\Delta \boldsymbol{v} \in L^{\frac{6}{5}}(\mathbb{R}^3)\ \ (\Rightarrow \boldsymbol{v} \in L^6(\mathbb{R}^3))$

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Remark. Since

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all (smooth enough) $\boldsymbol{v}$ satisfy GPG'94, $n \geq 4$.

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Therefore, $n = 3$ is the only case where an answer to Liouville's problem is not known.

Joseph Liouville

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THANK YOU!