# An Improved Energy Inequality for Weak Solutions of the Navier-Stokes Equations

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**Abstract**

Takeyuki Nagasawa proved in [3] the existence of a weak solution $u$ of the Navier-Stokes equations in a bounded domain with smooth boundary, which fulfils the improved energy inequality

$$\begin{aligned}\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 \mathrm{d}\tau + \limsup_{h\to 0} \int_h^t \frac{\|u(\tau)-u(\tau-h)\|_2^2}{h}\mathrm{d}\tau \\ \leq \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle u(\tau), f(\tau)\rangle_\Omega \mathrm{d}\tau\end{aligned} \tag{1}$$

for almost all $t \in (0,T)$. To prove the existence of such a weak solution he used a time discretisation method and the lower semicontinuity of all terms on the left-hand side of (1). Since we found a small gap in the proof of the lower semicontinuity of the third term in (1), we were not able to verify the correctness of his work. Under the assumption that his statement is valid, we proved the existence of a weak solution of the Navier-Stokes equations which fulfils the improved energy inequality (1) in a completely general domain. Furthermore, it can be shown that the Hausdorff dimension of all non-Lebesgue points of a weak solution of the Navier-Stokes equations which fulfils (1) is zero.

**Keywords:** improved energy inequality, Navier-Stokes equations, weak solution, partial regularity.

# References


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