Close-to-contact dynamics of solids inside a viscous fluid

M. Hillairet

Ceremade, Université Paris Dauphine

FSIforBIO 2011

Motivations

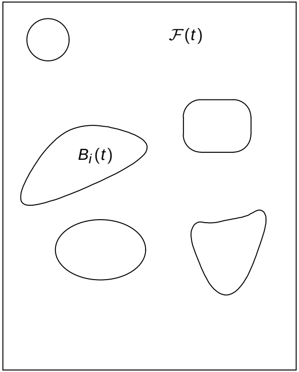

Figure 1. Rigid bodies in a fluid

Outline of the talks:

1. Classical equations
2. Study of the Cauchy problem in the viscous case
3. Close-to-contact motion of rigid bodies

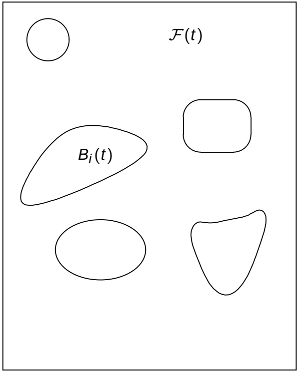

Figure 1. Rigid bodies in a fluid

Outline of the talks:

1. Classical equations
2. Study of the Cauchy problem in the viscous case
3. Close-to-contact motion of rigid bodies

Motivations :

- help numerical simulations
- find the limits of classical equations

- Classical equations

Solid description

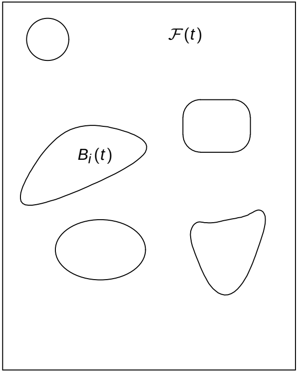

Figure 1. Rigid bodies in a fluid

Description of solid $\mathcal{B}_i$

Shape $B_i \subset\subset \mathbb{R}^n$ $(n = 2, 3)$:

$$\partial B_i \in \mathcal{C}^{1,1}, \quad \int_{B_i} \mathbf{x} \mathrm{d}\mathbf{x} = 0$$

Position $B_i(t)$:

$$B_i(t) = Q_i(t) B_i + \mathbf{G}_i(t),$$

where :

$$\mathbf{G}_i(t) \in \mathbb{R}^n, \; Q_i(t) Q_i^\top(t) = \mathbb{I}_n.$$

$$Q_i(0) = \mathbb{I}_n$$

Solid description

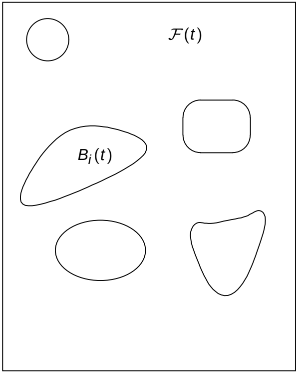

Figure 1. Rigid bodies in a fluid

Description of solid $\mathcal{B}_i$

Density $\rho_i \in (0,\infty)$

Mass $m_i \in (0,\infty)$,

$$m_i = \rho_i |B_i|$$

Inertia $\mathbb{J}_i$

$$\mathbf{a} \cdot \mathbb{J}_i \mathbf{b} = \rho_i \int_{B_i} [\mathbf{a} \times Q_i \mathbf{y}] \cdot [\mathbf{b} \times Q_i \mathbf{y}] \, \mathrm{d}\mathbf{y}.$$

Solid dynamics

Solid kinematics : Velocity-field

$$\mathbf{u}_i = \mathbf{V}_i + \omega_i \times (\mathbf{x} - \mathbf{G}_i),$$

where $\mathbf{V}_i = \dot{\mathbf{G}}_i$, and $\omega_i \times \text{"} = \text{"} \dot{Q}_i Q_i^\top$:

Solid dynamics.

(NL)
$$\left\{ \begin{aligned} m_i \frac{\mathrm{d}\mathbf{V}_i}{\mathrm{d}t} &= -\int_{\partial B_i(t)} \Sigma_i \mathbf{n}_i \, \mathrm{d}\sigma_i \\ \frac{\mathrm{d}[\mathbb{J}_i \omega_i]}{\mathrm{d}t} &= -\int_{\partial B_i(t)} [\mathbf{x} - \mathbf{G}_i] \times [\Sigma_i \mathbf{n}_i] \, \mathrm{d}\sigma_i \end{aligned} \right.$$

Notations : Σ_i stress tensor, $\mathbf{n}_i$ unit normal (toward $B_i(t)$).

Solid dynamics

Solid kinematics : Velocity-field

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where $\mathbf{V}_i = \dot{\mathbf{G}}_i$, and $\omega_i \times " = " \dot{Q}_i Q_i^\top$:

$$\omega \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \omega \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

Solid dynamics.

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$$\left\{ \begin{array}{rcl} m_i \dfrac{\mathrm{d}\mathbf{V}_i}{\mathrm{d}t} & = & -\displaystyle\int_{\partial B_i(t)} \Sigma_i \mathbf{n}_i \,\mathrm{d}\sigma_i \\ \dfrac{\mathrm{d}[\mathbb{J}_i \omega_i]}{\mathrm{d}t} & = & -\displaystyle\int_{\partial B_i(t)} [\mathbf{x} - \mathbf{G}_i] \times [\Sigma_i \mathbf{n}_i] \,\mathrm{d}\sigma_i \end{array} \right.$$

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Notations : Σ_i stress tensor, $\mathbf{n}_i$ unit normal (toward $B_i(t)$).

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_2 - y_2 x_1$$

Fluid equations

Fluid description

- density ρ, viscosities μ, λ
- velocity-field $\mathbf{u}$, pressure p

Fluid equations Compressible Navier Stokes equations

(CNS) $$\left\{ \begin{array}{rcl} \rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) & = & \mathrm{div}\,\Sigma + \rho\,\mathbf{f} \\ \partial_t \rho + \mathrm{div}\,(\rho\mathbf{u}) & = & 0 \\ p & = & a\rho^{\gamma} \end{array} \right. \quad \text{on } \mathcal{F}(t)$$

Fluid assumptions

- Compressible or Incompressible (constant-density)
- Inviscid or Viscous (newtonian)

$$\Sigma = \mu[\nabla\mathbf{u} + \nabla^{\top}\mathbf{u}] - (p + \lambda\mathrm{div}\mathbf{u})\mathbb{I}_n$$

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Fluid equations Incompressible Navier Stokes equations

$$\text{(INS)} \qquad \left\{ \begin{array}{rcl} \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) & = & \operatorname{div} \Sigma + \rho \mathbf{f} \\ \operatorname{div} \mathbf{u} & = & 0 \\ \rho & = & cstt \end{array} \right. \qquad \text{on } \mathcal{F}(t)$$

Fluid assumptions

- Compressible or Incompressible (constant-density)
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$$\Sigma = \mu[\nabla \mathbf{u} + \nabla^{\top} \mathbf{u}] - p\mathbb{I}_n$$

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Fluid equations Euler equations

$$(E) \qquad \left\{ \begin{array}{rcl} \rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) & = & \operatorname{div}\Sigma + \rho\,\mathbf{f} \\ \operatorname{div}\mathbf{u} & = & 0 \\ \rho & = & cstt \end{array} \right. \qquad \text{on } \mathcal{F}(t)$$

Fluid assumptions

- Compressible or Incompressible (constant-density)
- Inviscid or Viscous Newtonian

$$\Sigma = -p\mathbb{I}_n$$

Interactions

- The fluid domain is fixed by solid positions :

$$\mathcal{F}(t) = \Omega \setminus \overline{\bigcup_i B_i(t)}.$$

- Boundary conditions are fixed by solid dynamics:
  - Impermeability conditions :

    (BCi) $$(\mathbf{u} - \mathbf{u}_i) \cdot \mathbf{n}_i = 0 \text{ on } \partial B_i(t), \quad \mathbf{u} \cdot \mathbf{n}_0 = 0 \quad \text{on } \partial\Omega$$

  - No-slip conditions :

    (BCns) $$(\mathbf{u} - \mathbf{u}_i) \times \mathbf{n}_i = 0 \text{ on } \partial B_i(t), \quad \mathbf{u} \times \mathbf{n}_0 = 0 \quad \text{on } \partial\Omega$$

- Continuity of stress tensor :

$$\Sigma = \Sigma_i \quad \forall i.$$

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  - Slip conditions :

    (BCs) $$(\mathbf{u} - \mathbf{u}_i) \times \mathbf{n}_i = -\beta_i[\Sigma \mathbf{n}_i] \times \mathbf{n}_i \text{ on } \partial B_i(t), \quad \mathbf{u} \times \mathbf{n}_0 = -\beta_0[\Sigma \mathbf{n}_0] \times \mathbf{n}_0 \quad \text{on } \partial\Omega$$

- Continuity of stress tensor :

$$\Sigma = \Sigma_i \quad \forall i.$$

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II Construction of solutions

Full system (v-FSIS)

$N = 1 \quad \Omega = \mathbb{R}^2 \quad \mathbf{f} = 0 \quad \rho = 1$

$$\left\{ \begin{array}{rcl} (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) & = & \operatorname{div} \Sigma\,, \\ \operatorname{div} \mathbf{u} & = & 0\,, \end{array} \right. \qquad \forall\, (t,\mathbf{x}) \in \mathcal{Q}_T := \{(t,\mathbf{x})\,,\ \mathbf{x} \in \mathbb{R}^2 \setminus \overline{B}_1(t)\}\,,$$

$$\mathbf{u}(t,\mathbf{x}) = \mathbf{V}_1(t) + \omega_1(t) \times (\mathbf{x} - \mathbf{G}_1)\,, \quad \forall\, \mathbf{x} \in \partial B_1(t)\,, \qquad \lim_{|\mathbf{x}| \to \infty} \mathbf{u}(t,\mathbf{x}) = 0.$$

$$\left\{ \begin{array}{rcl} m_1 \dfrac{\mathrm{d}\mathbf{V}_1}{\mathrm{dt}} & = & -\displaystyle\int_{\partial B_1(t)} \Sigma \mathbf{n}_1 \,\mathrm{d}\sigma_1\,, \\ J_1 \dfrac{\mathrm{d}[\omega_1]}{\mathrm{dt}} & = & -\displaystyle\int_{\partial B_1(t)} [\mathbf{x} - \mathbf{G}_1] \times [\Sigma \mathbf{n}_1] \,\mathrm{d}\sigma_1\,, \end{array} \right.$$

$$\Sigma = \Sigma(\mathbf{u},p) := \mu(\nabla \mathbf{u} + \nabla^{\top} \mathbf{u}) - p\mathbb{I}_2\,,$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}^0\,, \quad \mathbf{G}_1(0) = \mathbf{G}_1^0\,, \quad \mathbf{V}_1(0) = \mathbf{V}_1^0\,, \quad \omega_1(0) = \omega_1^0\,, \quad Q_1(0) = \mathbb{I}_2.$$

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$$\mathbf{u}(0,\cdot) = \mathbf{u}^0 \,, \quad \mathbf{G}_1(0) = \mathbf{G}_1^0 \,, \quad \mathbf{V}_1(0) = \mathbf{V}_1^0 \,, \quad \omega_1(0) = \omega_1^0 \,, \quad Q_1(0) = \mathbb{I}_2.$$

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$$\mathbf{u}(t,\mathbf{x})=\mathbf{V}_1(t)+\omega_1(t)\times(\mathbf{x}-\mathbf{G}_1)\,,\quad \forall\,\mathbf{x}\in\partial B_1(t)\,,\qquad \lim_{|\mathbf{x}|\to\infty}\mathbf{u}(t,\mathbf{x})=0.$$

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Change of geometry

T. Takahashi '03, M. Tucsnak & T. Takahashi '04

$t \mapsto (\mathbf{V}_1, \omega_1)$ is given

Local velocity around the solid body :

$$\mathbf{u}_1(t, \mathbf{x}) = \mathbf{V}_1(t) + \omega_1(t) \times (\mathbf{x} - \mathbf{G}_1(t))$$

Change of geometry

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Local velocity around the solid body :

$$\mathbf{u}_1(t,\mathbf{x}) = \nabla^{\perp}\left[\omega_1(t)|\mathbf{x}-\mathbf{G}_1(t)|^2 - \mathbf{V}_1(t)\times(\mathbf{x}-\mathbf{G}_1(t)\right].$$

Change of geometry

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$t \mapsto (\mathbf{V}_1, \omega_1)$ is given

Local velocity around the solid body : $r = 2\,\mathrm{diam}(B_1)$

$$\text{(L)} \qquad \Lambda_1(t,\mathbf{x}) = \nabla^{\perp}\left[\chi_r(\mathbf{x} - \mathbf{G}_1^0)\left(\omega_1(t)|\mathbf{x} - \mathbf{G}_1(t)|^2 - \mathbf{V}_1(t) \times (\mathbf{x} - \mathbf{G}_1(t))\right)\right].$$

where

$$\chi_r(\mathbf{y}) = 1\,, \quad \forall\,|\mathbf{y}| < r\,, \qquad \chi(\mathbf{y}) = 0\,, \quad \forall\,|\mathbf{y}| > 2r\,.$$

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Local displacement around the solid body : X solution to

$$\text{(X)} \qquad \left\{\begin{array}{rcl} X(0,\mathbf{y}) & = & \mathbf{y}, \\ \dot{X}(t,\mathbf{y}) & = & \Lambda_1(t,X(t,\mathbf{y})). \end{array}\right. \qquad \forall\,\mathbf{y}\in\mathbb{R}^2.$$

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Remarks :

- $X(t,\cdot)$ is a smooth diffeomorphism
- $X(t,\mathbf{y}) = \mathbf{y}\,, \quad \forall\,|\mathbf{y}-\mathbf{G}_1^0| > 2r$
- $X = \mathbb{I}_2 + O(|\mathbf{V}_1| + |\omega_1|),$

Change of geometry

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Local velocity around the solid body : $r = 2\,\mathrm{diam}(B_1)$

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Remarks :

- $X(t,\cdot)$ is a smooth diffeomorphism
- $X(t,\mathbf{y}) = \mathbf{G}_1(t) + Q_1(t)\mathbf{y}\,,\quad \forall\,|\mathbf{y}-\mathbf{G}_1^0| < r - \|\mathbf{V}_1; L^1(0,t)\|$
- $X = \mathbb{I}_2 + O(|\mathbf{V}_1| + |\omega_1|)$,

Change of unknown

Given $(\mathbf{u}, p, \mathbf{G}_1, \mathbf{V}_1, \omega_1)$ such that

$$(\mathbf{V}_1, \omega_1) \in \mathcal{C}([0,T];\mathbb{R}^2) \times \mathcal{C}([0,T])$$

we introduce X and $Y = X^{-1}$ together with:

$$\overline{\mathbf{u}}(t,\mathbf{y}) = \nabla Y(t, X(t,\mathbf{y}))\mathbf{u}(t, X(t,\mathbf{y}))\,, \quad \overline{p}(t,\mathbf{y}) = p(t, X(t,\mathbf{y}))\,,$$
$$\overline{\mathbf{V}}_1(t) = Q_1^{\top}(t)\mathbf{V}_1(t)\,, \quad \overline{\omega}_1(t) = \omega_1(t)\,.$$

Definition

A collection $(\mathbf{u}, p, \mathbf{G}_1, \mathbf{V}_1, \omega_1)$ is a strong solution to (v-FSIS) on $(0,T)$ if

$$(\mathbf{V}_1, \omega_1) \in \mathbf{H}^1(0,T) \times H^1(0,T)\,, \quad \mathbf{G}_1(t) = \int_0^t \mathbf{V}_1(s)ds\,, \quad r > \|\mathbf{V}^1; L^1(0,T)\|,$$

if $\overline{\mathbf{u}} \in \mathcal{C}([0,T];\mathbf{H}^1(\mathcal{F}(0))) \cap L^2(0,T;\mathbf{H}^2(\mathcal{F}(0)))$, $\overline{p} \in L^2(0,T;H^1_{loc}(\mathcal{F}(0)))$
and $(\overline{\mathbf{u}}, \overline{p}, \overline{\mathbf{V}}_1, \overline{\omega}_1)$ satisfies the following system :

Problem in fixed geometry

$$\left\{\begin{array}{rcl} \partial_t \overline{\mathbf{u}} + [\,\mathbf{M}\overline{\mathbf{u}}\,] + [\,\mathbf{N}\overline{\mathbf{u}}\,] & = & [\,\mathbf{L}\overline{\mathbf{u}}\,] - [\,\mathbf{G}\overline{p}\,]\,, \\ \mathrm{div}\,\overline{\mathbf{u}} & = & 0\,, \end{array}\right. \qquad \text{in } \mathbb{R}^2 \setminus \overline{B}_1\,,$$

$$\overline{\mathbf{u}}(t,\mathbf{y}) = \overline{\mathbf{V}}_1(t) + \overline{\omega}_1(t) \times \mathbf{y}\,, \quad \forall\, \mathbf{y} \in \partial B_1\,, \qquad \lim_{|\mathbf{y}|\to\infty} \overline{\mathbf{u}}(t,\mathbf{y}) = 0.$$

$$\left\{\begin{array}{rcl} m_1 \dfrac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{V}}_1 & = & -\displaystyle\int_{\partial B_1} \Sigma(\overline{\mathbf{u}},\overline{p})\mathbf{n}_1 \mathrm{d}\sigma_1 - m_1 \overline{\omega}_1 \times \overline{\mathbf{V}}_1\,, \\ J_1 \dfrac{\mathrm{d}}{\mathrm{d}t}\overline{\omega}_1 & = & -\displaystyle\int_{\partial B_1} \mathbf{y} \times \Sigma(\overline{\mathbf{u}},\overline{p})\mathbf{n}_1 \mathrm{d}\sigma_1\,, \end{array}\right.$$

$$\overline{\mathbf{V}}_1(0) = \mathbf{V}_1^0\,, \qquad \overline{\omega}_1(0) = \omega_1^0, \quad \overline{\mathbf{u}}(0,\cdot) = \mathbf{u}^0\,,$$

where :

$$\begin{array}{rclrcl} [\,\mathbf{M}\overline{\mathbf{u}}\,] & \sim & \mathbf{u}\cdot\nabla\mathbf{u}\,, & [\,\mathbf{N}\overline{\mathbf{u}}\,] & \sim & \partial_t \mathbf{u} - \partial_t \overline{\mathbf{u}}, \\ [\,\mathbf{L}\overline{\mathbf{u}}\,] & \sim & \mu\Delta\mathbf{u} & [\,\mathbf{G}\overline{p}\,] & \sim & \nabla p\,. \end{array}$$

Problem in fixed geometry

$$\left\{\begin{array}{rcl} \partial_t \overline{\mathbf{u}} + [\,\mathbf{M}\overline{\mathbf{u}}\,] + [\,\mathbf{N}\overline{\mathbf{u}}\,] & = & [\,\mathbf{L}\overline{\mathbf{u}}\,] - [\,\mathbf{G}\overline{p}\,]\,, \\ \operatorname{div} \overline{\mathbf{u}} & = & 0\,, \end{array}\right. \qquad \text{in } \mathbb{R}^2 \setminus \overline{B}_1\,,$$

$$\overline{\mathbf{u}}(t,\mathbf{y}) = \overline{\mathbf{V}}_1(t) + \overline{\omega}_1(t) \times \mathbf{y}\,, \quad \forall\, \mathbf{y} \in \partial B_1\,, \qquad \lim_{|\mathbf{y}|\to\infty} \overline{\mathbf{u}}(t,\mathbf{y}) = 0.$$

$$\left\{\begin{array}{rcl} m_1 \dfrac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{V}}_1 & = & -\displaystyle\int_{\partial B_1} \Sigma(\overline{\mathbf{u}},\overline{p})\mathbf{n}_1 \mathrm{d}\sigma_1 - m_1 \overline{\omega}_1 \times \overline{\mathbf{V}}_1\,, \\ J_1 \dfrac{\mathrm{d}}{\mathrm{d}t}\overline{\omega}_1 & = & -\displaystyle\int_{\partial B_1} \mathbf{y} \times \Sigma(\overline{\mathbf{u}},\overline{p})\mathbf{n}_1 \mathrm{d}\sigma_1\,, \end{array}\right.$$

$$\overline{\mathbf{V}}_1(0) = \mathbf{V}_1^0\,, \qquad \overline{\omega}_1(0) = \omega_1^0, \quad \overline{\mathbf{u}}(0,\cdot) = \mathbf{u}^0\,,$$

where :

$$\begin{array}{rclrcl} [\,\mathbf{M}\overline{\mathbf{u}}\,] & = & o([\overline{\mathbf{u}},\overline{\mathbf{V}}_1,\overline{\omega}_1])\,, & [\,\mathbf{N}\overline{\mathbf{u}}\,] & = & o([\overline{\mathbf{u}},\overline{\mathbf{V}}_1,\overline{\omega}_1]), \\ [\,\mathbf{L}\overline{\mathbf{u}}\,] & = & \mu\Delta\overline{\mathbf{u}} + o([\overline{\mathbf{u}},\overline{\mathbf{V}}_1,\overline{\omega}_1]) & [\,\mathbf{G}\overline{p}\,] & = & \nabla\overline{p} + o([\overline{\mathbf{u}},\overline{\mathbf{V}}_1,\overline{\omega}_1])\,. \end{array}$$

Problem in fixed geometry

$$\begin{cases} \partial_t \overline{\mathbf{u}} & = & \mu\Delta\overline{\mathbf{u}} - \nabla\overline{p} + \mathbf{F}[\overline{\mathbf{u}},\overline{p},\overline{\mathbf{V}}_1,\overline{\omega}_1]\,, \\ \mathrm{div}\,\overline{\mathbf{u}} & = & 0\,, \end{cases} \qquad \text{in } \mathbb{R}^2 \setminus B_1\,,$$

$$\overline{\mathbf{u}}(t,\mathbf{y}) = \overline{\mathbf{V}}_1(t) + \overline{\omega}_1(t)\times\mathbf{y}\,,\quad \forall\,\mathbf{y}\in\partial B_1\,,\qquad \lim_{|\mathbf{y}|\to\infty}\overline{\mathbf{u}}(t,\mathbf{y}) = 0.$$

$$\begin{cases} m_1\dfrac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{V}}_1 & = & -\displaystyle\int_{\partial B_1}\Sigma(\overline{\mathbf{u}},\overline{p})\mathbf{n}_1\mathrm{d}\sigma_1 + \mathbf{F}_M[\overline{\mathbf{V}}_1,\overline{\omega}_1]\,, \\ J_1\dfrac{\mathrm{d}}{\mathrm{d}t}\overline{\omega}_1 & = & -\displaystyle\int_{\partial B_1}\mathbf{y}\times\Sigma(\overline{\mathbf{u}},\overline{p})\mathbf{n}_1\mathrm{d}\sigma_1 + T_M[\overline{\mathbf{V}}_1,\overline{\omega}_1].\,, \end{cases}$$

$$\overline{\mathbf{V}}_1(0) = \mathbf{V}_1^0\,,\qquad \overline{\omega}_1(0) = \omega_1^0,\quad \overline{\mathbf{u}}(0,\cdot) = \mathbf{u}^0\,,$$

where :

$$\begin{aligned} [\,\mathbf{M}\overline{\mathbf{u}}\,] &= o([\overline{\mathbf{u}},\overline{\mathbf{V}}_1,\overline{\omega}_1])\,, & [\,\mathbf{N}\overline{\mathbf{u}}\,] &= o([\overline{\mathbf{u}},\overline{\mathbf{V}}_1,\overline{\omega}_1]), \\ [\,\mathbf{L}\overline{\mathbf{u}}\,] &= \mu\Delta\overline{\mathbf{u}} + o([\overline{\mathbf{u}},\overline{\mathbf{V}}_1,\overline{\omega}_1]) & [\,\mathbf{G}\overline{p}\,] &= \nabla\overline{p} + o([\overline{\mathbf{u}},\overline{\mathbf{V}}_1,\overline{\omega}_1])\,. \end{aligned}$$

Strategy for solving the new problem

$$\left\{\begin{array}{rcl} \partial_t \mathbf{u} & = & \mu \Delta \mathbf{u} - \nabla p + \mathbf{F}\,, \\ \mathrm{div}\, \mathbf{u} & = & 0\,, \end{array}\right. \qquad \text{in } \mathbb{R}^2 \setminus B_1\,,$$

$$\mathbf{u}(t,\mathbf{y}) = \mathbf{V}_1(t) + \omega_1(t) \times \mathbf{y}\,, \quad \forall \mathbf{y} \in \partial B_1\,, \qquad \lim_{|\mathbf{y}| \to \infty} \mathbf{u}(t,\mathbf{y}) = 0.$$

$$\left\{\begin{array}{rcl} m_1 \dfrac{\mathrm{d}}{\mathrm{d}t} \mathbf{V}_1 & = & -\displaystyle\int_{\partial B_1} \Sigma(\mathbf{u},p)\mathbf{n}_1 \mathrm{d}\sigma_1 + \mathbf{F}_M\,, \\ J_1 \dfrac{\mathrm{d}}{\mathrm{d}t} \omega_1 & = & -\displaystyle\int_{\partial B_1} \mathbf{y} \times \Sigma(\mathbf{u},p)\mathbf{n}_1 \mathrm{d}\sigma_1 + T_M\,, \end{array}\right.$$

$$\mathbf{V}_1(0) = \mathbf{V}_1^0\,, \qquad \omega_1(0) = \omega_1^0, \quad \mathbf{u}(0,\cdot) = \mathbf{u}^0\,,$$

with polynomial nonlinearities :

$$\mathbf{F} \in \mathbf{L}^2((0,T) \times \mathcal{F}(0))\,, \quad \mathbf{F}_M \in \mathbf{L}^2(0,T)\,,\ T_M \in L^2(0,T)\,,$$

and initial data satisfying the compatibility conditions :

$$\mathbf{u}^0 \in \mathbf{H}^1(\mathbb{R}^2 \setminus \overline{B}_1\,,) \quad \mathrm{div}\, \mathbf{u}^0 = 0\,, \quad \mathbf{u}^0(\mathbf{y}) = \mathbf{V}_1^0 + \omega_1^0 \times \mathbf{y} \quad \forall \mathbf{y} \in \partial B_1.$$

Analysis of the linear problem

Lemma (M. Tucsnak& T. Takahashi '03)

Given

$$\mathbf{F} \in \mathbf{L}^2((0,T) \times \mathbb{R}^2 \setminus \overline{B}_1)\,, \quad \mathbf{F}_M \in \mathbf{L}^2(0,T)\,,\ T_M \in L^2(0,T)$$

and $(\mathbf{u}^0, \mathbf{V}^0, \omega^0) \in \mathbf{H}^1(\mathbb{R}^2 \setminus \overline{B}_1) \times \mathbb{R}^2 \times \mathbb{R}$ *s.t.*

$$div\, \mathbf{u}^0 = 0\,, \qquad \mathbf{u}^0(\mathbf{y}) = \mathbf{V}_1^0 + \omega_1^0 \times \mathbf{y}\,, \quad \forall \mathbf{y} \in \partial B_1\,,$$

there exists a unique solution $(\mathbf{u}, p, \mathbf{V}_1, \omega_1)$ *to the previous system such that*

$$\begin{aligned}
\mathbf{u} &\in \mathcal{C}([0,T]\,;\ \mathbf{H}^1(\mathbb{R}^2 \setminus \overline{B}_1)) \cap L^2(0,T;\mathbf{H}^2(\mathbb{R}^2 \setminus \overline{B}_1)) \cap H^1(0,T;\mathbf{L}^2(\mathbb{R}^2 \setminus \overline{B}_1)),\\
p &\in L^2(0,T;H^1_{loc}(\mathbb{R}^2 \setminus \overline{B}_1)),\\
\mathbf{V}_1 &\in H^1(0,T),\\
\omega_1 &\in H^1(0,T).
\end{aligned}$$

Moreover the mapping $(\mathbf{u}^0, \mathbf{V}^0, \omega^0, \mathbf{F}, \mathbf{F}_M, T_M) \mapsto (\mathbf{u}, \mathbf{V}_1, \omega_1)$ *is linear continuous.*

Introduction of a suitable operator

Functions spaces

$$\mathcal{H} = \{\mathbf{v} \in \mathbf{L}^2(\Omega), \text{div}\, \mathbf{v} = 0 \text{ in } \Omega,\ \exists (\mathbf{V}_\mathbf{v}, \omega_\mathbf{v}) \in \mathbb{R}^2 \times \mathbb{R}\,,\ \mathbf{v}(\mathbf{y}) = \mathbf{V}_\mathbf{v} + \omega_\mathbf{v} \times \mathbf{y}\,,\ \forall\, \mathbf{y} \in B_1\}.$$

$$\mathcal{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega), \text{div}\, \mathbf{v} = 0 \text{ in } \Omega,\ \exists (\mathbf{V}_\mathbf{v}, \omega_\mathbf{v}) \in \mathbb{R}^2 \times \mathbb{R}\,,\ \mathbf{v}(\mathbf{y}) = \mathbf{V}_\mathbf{v} + \omega_\mathbf{v} \times \mathbf{y}\,,\ \forall\, \mathbf{y} \in B_1\}.$$

Remark $\mathcal{H}$ is a closed subspace (projector $\mathbb{P}$) of $\mathbf{L}^2(\mathbb{R}^2)$ with the scalar product :

$$\begin{aligned}(\mathbf{v}, \mathbf{w}) &= \int_{\mathbb{R}^2 \setminus \overline{B}_1} \mathbf{u} \cdot \mathbf{w} + \rho_1 \int_{B_1} \mathbf{v} \cdot \mathbf{w}\,, \\ &= \int_{\mathbb{R}^2 \setminus \overline{B}_1} \mathbf{u} \cdot \mathbf{w} + m_1 \mathbf{V}_\mathbf{v} \cdot \mathbf{V}_\mathbf{w} + J \omega_\mathbf{v} \omega_\mathbf{w}.\end{aligned}$$

Operator definition $D(A) = \{\mathbf{v} \in \mathcal{V}, \mathbf{v} \in \mathbf{H}^2(\mathbb{R}^2 \setminus \overline{B}_1)\}\,,\ A = \mathbb{P}\mathcal{A}$:

$$\mathcal{A}\mathbf{v} = \begin{cases} \dfrac{\mu}{m_1} \displaystyle\int_{\partial B_1} 2D(\mathbf{u})\mathbf{n}_1 \mathrm{d}\sigma_1 + \left(\dfrac{\mu}{J_1} \displaystyle\int_{\partial B_1} \mathbf{z} \times (2D(\mathbf{u})\mathbf{n}_1) \mathrm{d}\sigma_1(\mathbf{z}) \right) \times \mathbf{y}\,, & \text{if } \mathbf{y} \in B_1\,, \\ -\mu \Delta \mathbf{u}\,, & \text{else.} \end{cases}$$

Analysis of A

Proposition

A is a maximal self-adjoint accretive operator.

Proof.

A is symmetric and positive. Let $(\mathbf{v},\mathbf{w}) \in D(A)^2$ then :

$$\begin{aligned}(A\mathbf{v},\mathbf{w}) &= -\mu\int_{\mathbb{R}^2\setminus\overline{B}_1} \Delta\mathbf{v}\cdot\mathbf{w} + \mu\int_{\partial B_1} D(\mathbf{v})\mathbf{n}_1 \mathrm{d}\sigma_1 \cdot \mathbf{V}_{\mathbf{w}} + \mu\omega_{\mathbf{w}}\int_{\partial B_1}[\mathbf{z}\times D(\mathbf{v})\mathbf{n}_1]\mathrm{d}\sigma_1 \\ &= 2\mu\int_{\mathbb{R}^2\setminus\overline{B}_1} D(\mathbf{v}) : D(\mathbf{w}) = \mu\int_{\mathbb{R}^2}\nabla\mathbf{v} : \nabla\mathbf{w}.\end{aligned}$$

A is maximal :

$$\forall \mathbf{f}\in\mathcal{H} \quad \exists \mathbf{v}\in D(A) \quad \mathbf{v} + A\mathbf{v} = \mathbf{f}, \text{ moreover } \|\mathbf{v}; \mathbf{H}^2(\mathbb{R}^2\setminus\overline{B}_1)\| \le \|\mathbf{f};\mathcal{H}\|.$$

- Existence and uniqueness : variational formulation + Lax Milgram theorem
- Regularity : Ellipticity estimates for Stokes system

□

Proof of **Lemma [M. Tucsnak& T. Takahashi '03]**

Let $(\mathbf{F}, \mathbf{F}_M, T_M)$ source term and $(\mathbf{u}^0, \mathbf{V}^0, \omega^0)$ initial data. Set $\widetilde{\mathbf{f}}$ and $\widetilde{\mathbf{u}}^0$

$$\widetilde{f}(t,\mathbf{y}) = \begin{cases} \dfrac{1}{m_1}\mathbf{F}_M(t) + \dfrac{1}{J_1}T_M(t) \times \mathbf{y} & \text{in } B_1 \\ \mathbf{F}(t,\mathbf{y}) & \text{else} \end{cases} \qquad \widetilde{\mathbf{u}}^0(\mathbf{y}) = \begin{cases} \mathbf{V}^0 + \omega^0 \times \mathbf{y} & \text{in } B_1 \\ \mathbf{u}^0(\mathbf{y}) & \text{else} \end{cases}$$

Construction of the candidate solution.

There exists a unique solution to $u' = Au + \mathbb{P}f$. such that:

$$u \in \mathcal{C}([0,T];\mathcal{V}) \cap L^2(0,T;D(A)) \cap H^1(0,T;\mathcal{H})$$

Set $\mathbf{u} = u_{|_{\mathbb{R}^2 \setminus \overline{B}_1}}$ $\quad \mathbf{V}_1 = \mathbf{V}_u \quad \omega_1 = \omega_u$.

$$\begin{aligned} \mathbf{u} &\in \mathcal{C}([0,T]\,;\, \mathbf{H}^1(\mathbb{R}^2 \setminus \overline{B}_1)) \cap H^1(0,T;\mathbf{L}^2(\mathbb{R}^2 \setminus \overline{B}_1)), \\ \mathbf{V}_1 &\in H^1(0,T), \\ \omega_1 &\in H^1(0,T). \end{aligned}$$

+ estimates.

Proof of **Lemma [M. Tucsnak & T. Takahashi '03]**

Verifications.

$\Longrightarrow$ div $\mathbf{u} = 0$ and boundary conditions O.K.

Given $\mathbf{w} \in \mathcal{C}_c^\infty(\mathbb{R}^2 \setminus \overline{B_1})$ s.t. $\operatorname{div} \mathbf{w} = 0$:

$$(u', \mathbf{w}) = -(Au, \mathbf{w}) + (\widetilde{f}, \mathbf{w})$$

Proof of **Lemma [M. Tucsnak & T. Takahashi '03]**

Verifications.

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Given $\mathbf{w} \in \mathcal{C}_c^\infty(\mathbb{R}^2 \setminus \overline{B}_1)$ s.t. $\operatorname{div} \mathbf{w} = 0$:

$$\int_{\mathbb{R}^2 \setminus \overline{B}_1} \partial_t \mathbf{u} \cdot \mathbf{w} = -(Au, \mathbf{w}) + (\widetilde{f}, \mathbf{w})$$

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$$\int_{\mathbb{R}^2 \setminus \overline{B}_1} \partial_t \mathbf{u} \cdot \mathbf{w} = \int_{\mathbb{R}^2 \setminus \overline{B}_1} \mu \Delta \mathbf{u} \cdot \mathbf{w} + (\widetilde{f}, \mathbf{w})$$

Proof of **Lemma [M. Tucsnak & T. Takahashi '03]**

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$\Longrightarrow$ there exists a pressure p s.t. $\partial_t \mathbf{u} = \mu \Delta \mathbf{u} - \nabla p + \mathbf{f}$
+ H^2 estimate

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\+ H^2 estimate

Given $\mathbf{w} \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ s.t. div $\mathbf{w} = 0$ and $\mathbf{w} = \mathbf{V}$ on B_1 : $(u', \mathbf{w}) + (Au, \mathbf{w}) = (\widetilde{f}, \mathbf{w})$

$$I_t + m_1 \dot{\mathbf{V}}_1 \cdot \mathbf{V} = I_\Delta - 2\mu \int_{\partial B} D(\mathbf{u})\mathbf{n}_1 \mathrm{d}\sigma_1 \cdot \mathbf{V} + I_f + \mathbf{F}_M \cdot \mathbf{V}$$

Proof of **Lemma [M. Tucsnak & T. Takahashi '03]**

Verifications.

$\Longrightarrow$ div $\mathbf{u} = 0$ and boundary conditions O.K.

Given $\mathbf{w} \in \mathcal{C}_c^\infty(\mathbb{R}^2 \setminus \overline{B}_1)$ s.t. $\operatorname{div} \mathbf{w} = 0$:

$$\int_{\mathbb{R}^2 \setminus \overline{B}_1} \partial_t \mathbf{u} \cdot \mathbf{w} = \int_{\mathbb{R}^2 \setminus \overline{B}_1} \mu \Delta \mathbf{u} \cdot \mathbf{w} + \int_{\mathbb{R}^2 \setminus \overline{B}_1} \mathbf{f} \cdot \mathbf{w}$$

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$$m_1 \dot{\mathbf{V}}_1 \cdot \mathbf{V} = \int_{B_1} \nabla p \cdot \mathbf{w} - 2\mu \int_{\partial B} D(\mathbf{u}) \mathbf{n}_1 \mathrm{d}\sigma_1 \cdot \mathbf{V} + \mathbf{F}_M \cdot \mathbf{V}$$

Proof of **Lemma [M. Tucsnak & T. Takahashi '03]**

Verifications.

$\Longrightarrow$ div $\mathbf{u} = 0$ and boundary conditions O.K.

Given $\mathbf{w} \in \mathcal{C}_c^\infty(\mathbb{R}^2 \setminus \overline{B}_1)$ s.t. $\operatorname{div} \mathbf{w} = 0$:

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Given $\mathbf{w} \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ s.t. $\operatorname{div} \mathbf{w} = 0$ and $\mathbf{w} = \mathbf{V}$ on B_1 : $(u', \mathbf{w}) + (Au, \mathbf{w}) = (\widetilde{f}, \mathbf{w})$

$$m_1 \dot{\mathbf{V}}_1 \cdot \mathbf{V} = -\int_{\partial B_1} \Sigma(\mathbf{u}, p) \mathbf{n}_1 d\sigma_1 \cdot \mathbf{V} + \mathbf{F}_M \cdot \mathbf{V}$$

Proof of **Lemma [M. Tucsnak & T. Takahashi '03]**

Verifications.

$\Longrightarrow$ div $\mathbf{u} = 0$ and boundary conditions O.K.

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$$m_1 \dot{\mathbf{V}}_1 \cdot \mathbf{V} = -\int_{\partial B_1} \Sigma(\mathbf{u}, p)\mathbf{n}_1 \mathrm{d}\sigma_1 \cdot \mathbf{V} + \mathbf{F}_M \cdot \mathbf{V}$$

$\Longrightarrow$ Newton laws.

Local existence result

Theorem ($N = 1, \Omega = \mathbb{R}^2$)
Given initial data $(\mathbf{u}^0, (\mathbf{G}_1^0, \mathbf{V}_1^0, \omega_1^0)) \in \mathbf{H}^1(\mathcal{F}(0)) \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ *satisfying*

$$div\, \mathbf{u}^0 = 0 \quad \forall\, \mathbf{x} \in \mathcal{F}(0)\,, \qquad \mathbf{u}^0(\mathbf{x}) = \mathbf{V}_1^0 + \omega_1^0 \times (\mathbf{x} - \mathbf{G}_1^0) \qquad \forall\, \mathbf{x} \in \partial B_1(0)\,,$$

there exists $T_0 > 0$ *depending only on the size of initial data such that there exists a unique strong solution to (v-FSIS) on* $(0, T_0)$ *with the given initial data.*

Local existence result

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$$div\, \mathbf{u}^0 = 0 \quad \forall \mathbf{x} \in \mathcal{F}(0)\,, \qquad \mathbf{u}^0(\mathbf{x}) = \mathbf{V}_1^0 + \omega_1^0 \times (\mathbf{x} - \mathbf{G}_1^0) \qquad \forall \mathbf{x} \in \partial B_1(0)\,,$$

there exists $T_0 > 0$ *depending only on the size of initial data such that there exists a unique strong solution to (v-FSIS) on* $(0, T_0)$ *with the given initial data.*

Theorem (general case)

Given a source term $\mathbf{f} \in L^2_{loc}(0,\infty; W^{1,\infty}(\Omega) \cap L^2(\Omega))$, *we assume initial data* $(\mathbf{u}^0, (\mathbf{G}_i^0, \mathbf{V}_i^0, \omega_i^0)_{i\in\{1,\dots,N\}})$ *satisfy a no-contact assumption and*

$$\mathbf{u}^0 \in \mathbf{H}^1(\mathcal{F}(0)) \qquad div\, \mathbf{u}^0 = 0 \quad \forall \mathbf{x} \in \mathcal{F}(0)\,,$$

$$\mathbf{u}^0(\mathbf{x}) = \mathbf{V}_i^0 + \omega_i^0 \times (\mathbf{x} - \mathbf{G}_i^0) \qquad \forall \mathbf{x} \in \partial B_i(0) \quad \forall i \in \{1, \dots, N\}.$$

Then, there exists $T_0 > 0$ *depending only on the size of initial data and the initial configuration such that there exists a unique strong solution to (v-FSIS) on* $(0, T_0)$ *with the given initial data.*

Case $N = 1$, $\Omega = \mathbb{R}^2$

Theorem (Blow-up alternative)

Given $(\mathbf{u}^0, (\mathbf{G}_1^0, \mathbf{V}_1^0, \omega_1^0)) \in \mathbf{H}^1(\mathcal{F}(0)) \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ *satisfying*

$$div\, \mathbf{u}^0 = 0 \quad \forall \mathbf{x} \in \mathcal{F}(0)\,, \qquad \mathbf{u}^0(\mathbf{x}) = \mathbf{V}_1^0 + \omega_1^0 \times (\mathbf{x} - \mathbf{G}_1^0)\,, \qquad \forall \mathbf{x} \in \partial B_1(0)\,,$$

there exists a unique maximal strong solution to (v-FSIS) with initial data $(\mathbf{u}^0, \mathbf{G}_1^0, \mathbf{V}_1^0, \omega_1^0)$. *Moreover, the existence time* T_* *of the maximal solution satisfies the alternative :*

- *either* $T_* = \infty$
- *either* $T_* < \infty$ *and* $\limsup_{t \to T_*} \|\mathbf{u}(t,\cdot)\ ;\ H^1(\mathbb{R}^2 \setminus \overline{B}_1(t))\| = \infty$.

First order estimate. Multiply momentum equation by $\mathbf{u}$.

$$\int_{\mathcal{F}(t)} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \int_{\mathcal{F}(t)} \mathrm{div}\Sigma(\mathbf{u}, p) \cdot \mathbf{u}$$

Case $N = 1$, $\Omega = \mathbb{R}^2$

Theorem (Blow-up alternative)

Given $(\mathbf{u}^0, (\mathbf{G}_1^0, \mathbf{V}_1^0, \omega_1^0)) \in \mathbf{H}^1(\mathcal{F}(0)) \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ *satisfying*

$$div\, \mathbf{u}^0 = 0 \quad \forall \mathbf{x} \in \mathcal{F}(0)\,, \qquad \mathbf{u}^0(\mathbf{x}) = \mathbf{V}_1^0 + \omega_1^0 \times (\mathbf{x} - \mathbf{G}_1^0)\,, \qquad \forall \mathbf{x} \in \partial B_1(0)\,,$$

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Case $N = 1$, $\Omega = \mathbb{R}^2$

Theorem (Blow-up alternative)

Given $(\mathbf{u}^0, (\mathbf{G}_1^0, \mathbf{V}_1^0, \omega_1^0)) \in \mathbf{H}^1(\mathcal{F}(0)) \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ *satisfying*

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Formal estimates

$B_1 = B(0, R)$

Assume $(\mathbf{u}, p, \mathbf{G}_1, \mathbf{V}_1, \omega_1)$ is a solution to (v-FSIS)

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$$\frac{3}{4}\int_{\mathcal{F}(t)}|\partial_t\mathbf{u}|^2 - C\|\mathbf{u}\cdot\nabla\mathbf{u}; L^2(\mathcal{F}(t))\|^2$$

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$$\sup_{(0,T)} \|\mathbf{u}^0; H^1(\mathcal{F}(t))\| \leq C(\mathbf{u}_0, \mathbf{V}_1^0, \omega_1^0, T).$$

Case $N \geq 1$ and $\Omega \subset\subset \mathbb{R}^2$

With a similar construction, it yields :

Theorem

Given a source term $\mathbf{f} \in L^2_{loc}(0,\infty; W^{1,\infty}(\Omega) \cap L^2(\Omega))$, *we assume initial data* $(\mathbf{u}^0, (\mathbf{G}_i^0, \mathbf{V}_i^0, \omega_i^0)_{i\in\{1,\dots,N\}})$ *satisfying a no-contact assumption and*

$$\mathbf{u}^0 \in \mathbf{H}^1(\mathcal{F}(0)), \qquad div\, \mathbf{u}^0 = 0, \quad \forall \mathbf{x} \in \mathcal{F}(0),$$

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Then, there exists a maximal T_* *such that there exists a unique strong solution to (v-FSIS) on* $(0, T_*)$ *with the given initial data. Moreover, there holds the alternative*

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Question : Does contact mean a "real" blow-up ?